Infrared behavior of Weyl Gravity and Relativistic Dynamics

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International Association of Relativistic Dynamics - Conference
IARD 2020, Czech Technical University (CTU), Prague

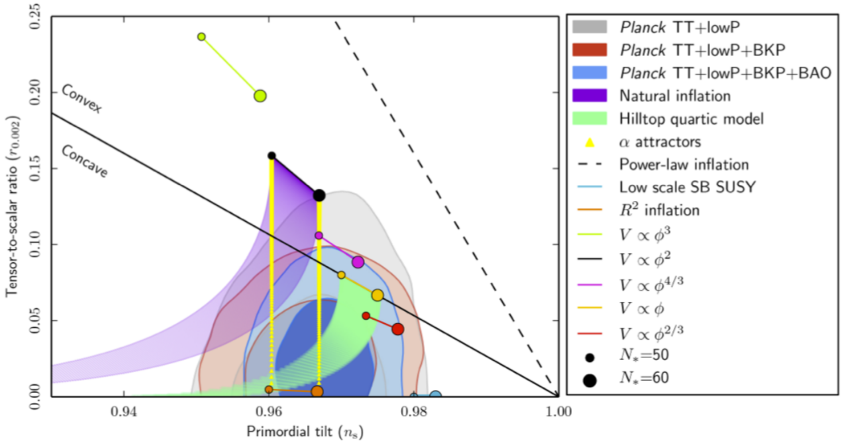
General Relativity and Astrophysics - Online Session

1st of June 2020

This talk is based on collaboration with prof. P. Jizba (CTU)

and financed by European Union,
under the grant "Investigations of conformal symmetry in gravity and particle physics"
and based on the paper **PRD 101, 044050 (2020)**; 1912.10271

Why conformal gravity

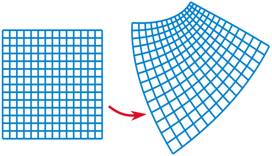


$$n_s = 1.00 - \varepsilon, \quad \varepsilon \ll 1$$

Why conformal gravity

Conformal symmetry and conformal transformations

- preserves only angles, distances are relative (gauge dependent)
- touches the differential structure of the spacetime
- leaves invariant only the causal structure of spacetime
- present for the classical matter system at very high energy or when particles are effectively massless
- can be additional symmetry of the gravitational system
- can be treated like gauge symmetry (can be gauged)



Advantages of conformal symmetry

Quantum conformality

- means that quantum fluctuations look the same at all energy scales (scale-invariance)
- additional symmetry on the quantum level
- constrains further quantum dynamics
- may be helpful in avoiding divergences
- leads to CFT's which describe FP's of RG
- gives fully idempotent quantum effective action Γ
- gives rise to very well-behaved quantum models, like $\mathcal{N} = 4$ SYM theory or $\mathcal{N} = 4$ conformal supergravity of Fradkin and Tseytlin '85
- may be instrumental in solving the issue of spacetime singularities
- is a starting point for various (non-conformal) deformations

Weyl gravity

Classical conformal Weyl gravity in $d = 4$

- is diffeomorphicaly invariant
- is conformally invariant
- defined in $d = 4$ by the action

$$S_{\rm conf} = \int d^4x \sqrt{|g|} \alpha_C C^2,$$

where

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2R_{[\mu[\rho}g_{\sigma]\nu]} + \frac{1}{3} g_{[\mu\rho}g_{\nu]\sigma} R$$

is a Weyl tensor (tensor of conformal curvature)

- uniqueness – defined by only one gravitational coupling constant: $\alpha_C > 0$

Weyl gravity

Advantages of conformal Weyl gravity in $d = 4$

- provides explanation for "almost" scale-invariant power spectrum of cosmological fluctuations
- describes accurately > 100 galactic rotation curves ("dark matter" problem)
- all vacuum solutions of Einsteinian gravitation are also exact vacuum solutions here (like Schwarzschild, Kerr spacetimes etc.)
- reproduces all precise tests of relativistic gravitation like Einstein theory in the vacuum
- resolves problem of classical singularities of GR (of Big Bang, or black hole) by allowing for conformal rescalings:

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x)' = \Omega^2(x) g_{\mu\nu}(x)$$

Quantum Weyl gravity

Classical EM and YM (non-Abelian) gauge theories are conformally invariant in $d = 4$

Advantages of quantum conformal Weyl gravity in $d = 4$

- allows coupling of conformal matter in a conformal symmetry preserving way
- all the divergences from the matter sector (of the type C^2) are absorbed – conformal deWitt-Utiyama argument
- is a renormalizable quantum theory of gravitation
- is an asymptotically free theory in the UV regime (like QCD)
- reaches an UV FP of RG
- for $\mathcal{N} = 4$ super version is UV-finite (no divergences!)
- lets performance of well controlled quantum computations

Quantum Weyl gravity

Advantages of quantum conformal Weyl gravity in $d = 4$

- this approach to relativistic gravitation is fully Machian, gravitational field here is determined completely by the matter distribution everywhere else in the Universe
- for quantum conformality no conformal (gauged) anomaly on the quantum level, theory is with quantum scale-invariance, no beta functions, no RG flow, sits at the the FP of RG

Quantum Weyl gravity in $d = 4$ in Euclidean framework

- C^2 action is bounded from below, no conformal instability problem
- positive-definiteness of the partition function Z
- gives a consistent quantum statistical mechanics of 4-dimensional differential manifolds (quantum shape dynamics)

Relativistic dynamics

Quantum Weyl Conformal Gravity

- is a system with very peculiar relativistic dynamics
- diffeomorphism symmetry (coordinate change) Diff is fully realized
- besides Diff also the conformal symmetry is gauged (local version)

- is a first modified gravitational theory (introduced by Weyl in 1918)
- different than Einsteinian gravity (introduced in 1915)

- is a very constrained relativistic system of gravitational field
- Veltman discontinuity of HD gravitational theories: $\alpha_{R^2} \to 0$ does not work
- physical degrees of freedom are different than in generic HD theories: only transverse traceless (TT) gravitons

Quantum computations

Examples of quantum computations in Weyl Conformal Gravity

- within QFT of gravitational (and conformal) interactions
- within background-independent formalism (BFM)
- using only physical degrees of freedom (TT gravitons)

1. 1-loop partition function
2. functionally improved RG beta functions
3. quest for IR FP $\Longrightarrow$ windows for cosmology

York decomposition

York decomposition of gravitational fluctuations

- traceless

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \phi$$

$$\bar{h}^{\mu}_{\mu} = 0, \quad \phi = h^{\mu}_{\mu}$$

- transverse and traceless (TT)

$$\bar{h}_{\mu\nu} = \bar{h}^{\perp}_{\mu\nu} + \nabla_{\mu} \eta^{\perp}_{\nu} + \nabla_{\nu} \eta^{\perp}_{\mu} + \nabla_{\mu} \nabla_{\nu} \sigma - \frac{1}{4} g_{\mu\nu} \Box \sigma$$

$$\nabla_{\mu} \bar{h}^{\perp}_{\mu\nu} = 0, \quad \nabla_{\mu} \eta^{\perp}_{\mu} = 0, \quad \bar{h}^{\perp\mu}_{\ \ \mu} = 0 \quad \text{with} \quad \bar{\phi} = \phi - \Box \sigma$$

- the physical field in conformal gravity is only TT graviton field $\bar{h}^{\perp}_{\mu\nu} = h^{TT}_{\mu\nu}$ due to diffeomorphism and conformal symmetry

Invariants in QG

Local curvature invariants in $d = 4$ of dimension four

- conformal invariant C^2 :

$$C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$$

  $\delta_c(\sqrt{|g|}C^2) = 0$

- topological invariant GB :

$$\mathrm{GB} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

  $\delta\left(\int d^4x\sqrt{|g|}\mathrm{GB}\right) = 0$

- globally scale-invariant R^2:
  $\mathrm{dim}(R^2) = 4$

Background Field Method

Background manifolds

- maximally symmetric spaces (MSS)

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \quad \text{with} \quad \Lambda = \text{const}$$

  where also $R_{\mu\nu} = \Lambda g_{\mu\nu}$, $R = 4\Lambda$, $\mathrm{GB} = \frac{8}{3}\Lambda^2$ and $C_{\mu\nu\rho\sigma} = 0$, $C^2 = 0$
- Ricci-flat spaces

$$R_{\mu\nu} = 0,\; R = 0,\; R^2_{\mu\nu\rho\sigma} = C^2 = \mathrm{GB}$$

- Einstein spaces (ES)

$$R_{\mu\nu} = \Lambda g_{\mu\nu},\; R = d\Lambda,\; \Lambda = \text{const}$$

  where also $C^2 = R^2_{\mu\nu\rho\sigma} - \frac{8}{3}\Lambda^2$, $\mathrm{GB} = R^2_{\mu\nu\rho\sigma}$ and $\mathrm{GB} = C^2 + \frac{8}{3}\Lambda^2$
- they are all Bach-flat in $d = 4$ dimensions (classical solutions of conformal Weyl gravity)

Partition function on MSS

Derivation by Fradkin and Tseytlin '85

- second variation around MSS background in TT fluctuation fields

$$\delta^2 S = \int d^4x \sqrt{g} h_{\mu\nu}^{TT} \left(\Box - \frac{2}{3}\Lambda \right) \left(\Box - \frac{4}{3}\Lambda \right) h_{\mu\nu}^{TT}$$

- after taking Jacobian of the change of variables under PI and the FP determinant into account

$$Z_{1-\text{loop}}^2 \sim \det{}^{-1} \left(\frac{\delta^2 S}{\delta h_{\mu\nu}^2} \right) = \frac{\det_{1T}\left(\Box + \Lambda\right) \det_0 \left(\Box + \frac{4}{3}\Lambda\right)}{\det_{2TT}\left(\Box - \frac{2}{3}\Lambda\right) \det_{2TT}\left(\Box - \frac{4}{3}\Lambda\right)}$$

- correcting by contribution of zero modes

$$Z_{1-\text{loop}}^2 = \frac{\det_1^2\left(\Box + \Lambda\right) \det_1\left(\Box + \frac{1}{3}\Lambda\right) \det_0\left(\Box + \frac{4}{3}\Lambda\right)}{\det_{2T}\left(\Box - \frac{2}{3}\Lambda\right) \det_{2T}\left(\Box - \frac{4}{3}\Lambda\right) \det_0\left(\Box + 2\Lambda\right)}$$

Partition function on Ricci-flat background

Derivation by Fradkin and Tseytlin '85

- second variation around Ricci-flat background in TT fluctuation fields

$$\delta^2 S = \int d^4x \sqrt{g} h_{\mu\nu}^{TT} \left(\Box - 2\hat{C}\right)^2 h_{\mu\nu}^{TT}$$

- after taking Jacobian of the change of variables under PI and the FP determinant into account

$$Z_{1-\mathrm{loop}}^2 \sim \det{}^{-1}\left(\frac{\delta^2 S}{\delta h_{\mu\nu}^2}\right) = \frac{\det_1^3 \Box \, \det_0^2 \Box}{\det_2^2 \left(\Box - 2\hat{C}\right)}$$

- there is no correcting contribution from zero modes

Functional RG flow

RG flow equation

$$\partial_t \Gamma_k = \mathrm{Tr}\left(\frac{\partial_t R_k \hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}} + a \hat{\mathbf{1}}} \right)$$

Truncation ansatz

For the L.H.S. of the flow equation

$$k \frac{d}{dk} \Gamma_k^L = \beta_C C^2 + \beta_{\mathrm{GB}} \mathrm{GB}$$

at the one-loop level we have $\beta_R R^2 = 0$ in Weyl conformal gravity (Fradkin Tseytlin '85). For the R.H.S. $\Gamma_k^R = C^2$

Two beta functions β_C and β_{GB}

available by projecting the flow to MSS or Ricci-flat backgrounds (or ES in one stroke)

Flow on MSS

$$\partial_t \Gamma_k^L = \mathrm{Tr}_{2T}\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} - \frac{2}{3}\Lambda\hat{\mathbf{1}}}\right) + \mathrm{Tr}_{2T}\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} - \frac{4}{3}\Lambda\hat{\mathbf{1}}}\right) - 2\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \Lambda\hat{\mathbf{1}}}\right) -$$

$$-\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \frac{1}{3}\Lambda\hat{\mathbf{1}}}\right) + \mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + 2\Lambda\hat{\mathbf{1}}}\right) - \mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \frac{4}{3}\Lambda\hat{\mathbf{1}}}\right)$$

from which we derive

$$\beta_{\mathrm{GB}} = \frac{1}{2}(2-\eta)\left[-\frac{21}{40}\left(1-\frac{\frac{2}{3}\Lambda}{k^2}\right)^{-1} + \frac{9}{40}\left(1-\frac{\frac{4}{3}\Lambda}{k^2}\right)^{-1} -\right.$$

$$\left.-\frac{179}{45}\left(1+\frac{\Lambda}{k^2}\right)^{-1} - \frac{59}{90}\left(1+\frac{\frac{1}{3}\Lambda}{k^2}\right)^{-1} + \frac{479}{360}\left(1+\frac{2\Lambda}{k^2}\right)^{-1} - \frac{269}{360}\left(1+\frac{\frac{4}{3}\Lambda}{k^2}\right)^{-1}\right]$$

- consistent with the one-loop results in Weyl gravity by Fradkin and Tseytlin, if $\eta = 0$ and $k \to \infty$, i.e. $\beta_{\mathrm{GB}} = -\frac{87}{20}$

Flow on Ricci-flat background

$$\partial_t \Gamma_k^L = 2\mathrm{Tr}_2\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} - 2\hat{C} + R_k \hat{\mathbf{1}}}\right) - 3\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}}}\right) - 2\mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}}}\right)$$

from which we derive

$$\partial_t \Gamma_k^L = \frac{1}{2}(2-\eta)\int d^4x\sqrt{g}\left(2\frac{19}{18}C^2 - 3\left(-\frac{11}{180}\right)C^2 - 2\left(\frac{1}{180}\right)C^2\right) =$$

$$= (2-\eta)\int d^4x\sqrt{g}\left(\frac{411}{180}C^2\right)$$

$$\beta_C + \beta_{\mathrm{GB}} = \frac{1}{2}(2-\eta)\left(\frac{137}{60}\right)$$

- consistent with the one-loop results in Weyl gravity by Fradkin and Tseytlin, if $\eta = 0$, i.e. $\beta_C + \beta_{\mathrm{GB}} = \frac{137}{60} \quad \Longrightarrow \quad \beta_C = \frac{199}{30}$

Beta functions and fixed points

Beta functions

- the system for two beta functions can be solved algebraically
- anomalous dimension of the conformal graviton field

$$\eta = -\frac{1}{\omega_C}\partial_t \omega_C$$

can be included for free

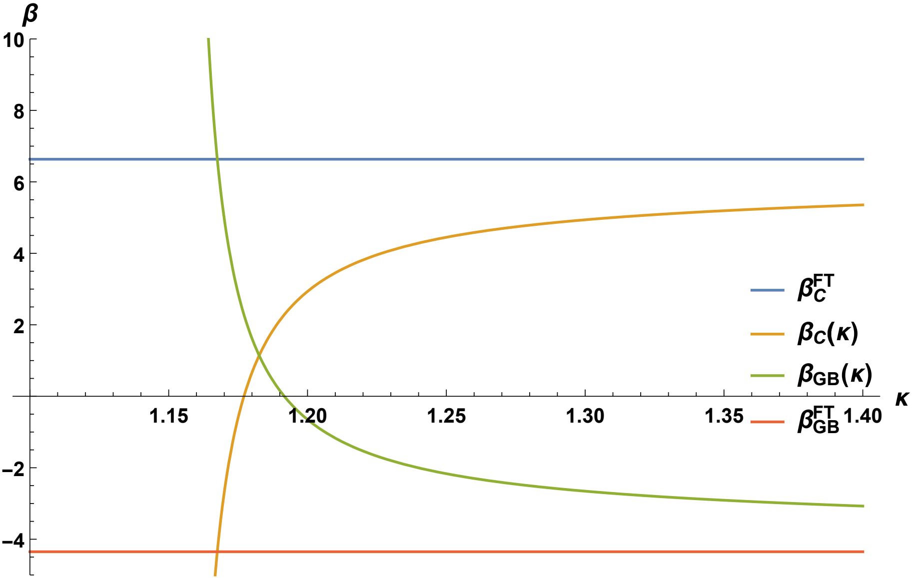
FP's

- UV: for $k \to +\infty$ asymptotic freedom in the coupling $\left(\sqrt{\omega_C}\right)^{-1}$ and $\left(\sqrt{-\omega_{\rm GB}}\right)^{-1}$, Gaussian FP
- IR: for $\beta_C = 0$ and $\beta_{\rm GB} = 0$ conditions we find respectively

$\kappa_C \approx 1.17709 \quad \text{and} \quad \kappa_{\rm GB} \approx 1.19163$ for the case of $\Lambda > 0$
$\kappa_C \approx 1.49722 \quad \text{and} \quad \kappa_{\rm GB} \approx 1.52128$ for the case of $\Lambda < 0$

with the rescaling $\kappa = \frac{k}{\sqrt{|\Lambda|}}$; 2% confluence of FP's

Beta functions for $\Lambda > 0$



IR FP

Turning point (TP)

- two lines cross zero line almost at the same energy scale
- this is however a turning point of RG flow
- it corresponds to multi-branch holographic RG flow
- using AdS/CFT: bounce solutions for FLRW scale factor in the 5d bulk

Towards (true) IR FP

- Perturbation calculus in couplings at the TP ω_*
- treating turning point as the start for perturbation
- non-trivial IR FP is found with

$$\omega_{**} = \omega_* + \frac{9}{2}\kappa\beta'(\kappa)$$

- anomalous dimensions of two operators $\theta = \left\{-\frac{1}{3}, -\frac{1}{3}\right\}$
- the IR FP is completely IR–stable

Bouncing dynamics & Holography

Turning point (TP)

- happens at some finite energy scale $\kappa = \kappa_c$
- corresponds to a 4d surface embedded in AdS-like $d = 5$ geometry located at some finite radial coordinate $\rho_c \sim \kappa_c^{-1}$
- But true IR FP shall correspond to a conformal boundary of AdS at infinite values of the radial AdS coordinate

Gravitational bounce

- example of an FLRW spacetime for which the cosmological scale factor $a(t)$ exhibits a bounce behavior
- some (exotic) matter source present on the brane located at $\kappa = \kappa_c$
- profile of the bulk scalar field with square-root like singularity and a non-analytic behaviour
- critical point is a joining point (or a bifurcation point) for two branches of real solutions for the bulk profile of the scalar field

Conclusions

Final comments

- The IR FP is for finite values of energies (moreover Λ-dependent), but can be shifted into the deep IR, where formally $k = 0$
- The IR FP is entirely due to threshold phenomena, and the inclusion of the anomalous dimension η (even with a supposed exact non-perturbative expression) does not change anything for the existence of it
- The FP is for any values of the couplings $\omega_{\rm GB}$ and ω_C.
  There is no any constraint from which it could be possible to find some special FP values of the couplings $\omega_{\rm GB}^*$ and ω_C^*.
  It is a 2-dimensional surface of FP's

Physical features in cosmology

- UV FP – Big Bang
- IR FP – onset of inflation
- scale Λ – radius of the inflationary de Sitter background, when $\Lambda > 0$

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Děkuji!

Thank you!

Unitarity issue

Possible solutions

- perturbative: Lee-Wick-Anselmi modification of Feynman prescription of avoiding poles of the propagator
- numerical: ghosts do not show up at low energies (below Planck scale) (Shapiro et al.)
- non-perturbative:
  - quantization within PT-symmetric form of QM (Bender, Mannheim),
  - perturbation around wrong vacuum (instabilities like with tachyons), another gravitational vacuum (MSS) should be used instead,
  - unitarity is safe due to non-trivial UV FP (asymptotic safety),
  - conformal supergravity (Fradkin, Tseytlin) is so special that the problem is solved thanks to supersymmetry and conformal symmetries (in $\mathcal{N} = 4$ SYM there is no problem with unitarity),
  - in true quantum CFT there are no scattering amplitudes