# Wave-particle duality in general relativity

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# Outline

# Abstract

In previous work, the Hamiltonian Jacobi equation has been associated with the metrics of general relativity and shown to be a generalized Dirac equation for quantum mechanics. This theory is now further devoloped to show its implications for the Zitterbewugung problem and its relationship to isotropy. Moreover, it is also shown that for the theory to be consistent, the momentum defined by the Hamilton-Jacobi function presupposes the existence of a universal paramenter in agreement with the Horwitz-Stueckeleberg theory. In the case of particles with mass this invariant parameter can be defined by $d\lambda = dt/m(t)$ where $t$ has the units of time and $m = m(t)$ has the units of mass. For examples, for a particle trajectory parametrized by the proper time $d\tau$, the Hamilton-Jacobi formalism requires that $d\lambda = d\tau/m_o$ , where $m_0$ is the rest mass.

# Wave-Particle Duality

- Linearized Metrics in Tetrad Coordinates
- Let $\mathbf{ds} = (dx^\mu)$ be a four vector of length $ds$ such that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b$$

- The equivalent linear spinor representation is given by

$$\tilde{ds} \equiv \gamma_a dx^a,$$

where $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ and $ds^2 = \tilde{ds}^2$ a scalar.

- Define a corresponding eigenvector equation with eigenvalue $ds$ by

$$\tilde{ds}\xi = ds\xi \tag{1}$$

- The $\tilde{ds}$ matrix can be considered as the dual of the expression

$$\tilde{\partial}_s \equiv \gamma^a \frac{\partial}{\partial x^a}$$

which operates on spinor functions $\psi \in L^2$ space.

- Multiplying both expressions together we obtain the Clifford product:

$$\frac{\tilde{ds}}{d\lambda}.\tilde{\partial}_s\psi = \frac{1}{2}\left\{\frac{\tilde{ds}}{d\lambda}, \tilde{\partial}_s\psi\right\} + \frac{1}{2}\left[\frac{\tilde{ds}}{d\lambda}, \tilde{\partial}_s\psi\right] \quad (2)$$

$$= \frac{d\psi}{d\lambda} + \frac{\vec{ds}}{d\lambda} \wedge \nabla\psi \ . \quad (3)$$

- Note

$$\frac{d\psi}{d\lambda} = \frac{\partial\psi}{\partial x^a}\frac{dx^a}{d\lambda}$$

defines a Hamilton-Jacobi equation with $p_a = \frac{\partial\psi}{\partial x^a}$ where $p_0 = -H$

- $\lambda$ can be considered as the SHP universal (time??) parameter.

# Wave-particle duality and Hamilton-Jacobi Functions

- A Special Theorem

- Lemma: Let $F(\mathbf{x}, t)$ be a function and $\psi(F) = \exp(kF)$,$k$ constant then $p^a = \eta^{ab}\frac{\partial F}{\partial x_i^b}$ iff $kp^a\psi = \eta^{ab}\frac{\partial \psi}{\partial x_i^b}$.

  **Proof:** Trivial. It is sufficient to substitute.

- Corollary: If $k = 1$ and $F = W = \int \eta^{ab} p_a dx_b = \int Hdt - \mathbf{pdx}$ is the Hamilton-Jacobi (HJ) function then $\psi(W)$ is also a HJ function such that

$$\gamma^a \frac{\partial \psi}{\partial x^a} = \gamma^a p_a \psi.$$

- Indeed, heuristically, the Hamilton-Jacobi function can be directly related to the metric expressed locally in tetrad coordinates as follows:

$$W = \int mc\frac{ds}{dt}ds = \int mcdt - \mathbf{pdx}, \qquad \text{where} \qquad m = m_0\frac{dt}{d\tau}, \quad (4)$$

where $m = m_0 \frac{dt}{d\tau}$ and $\mathbf{p} = m\frac{\mathbf{dx}}{dt}$ .

# A choice of SHP parameter $\lambda$

- There are two (local) invariants

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \qquad d\psi = \frac{\partial \psi}{\partial x^a} dx^a = \frac{d\psi}{ds} ds$$

- These can be related to each other by defining a Hamilton-Jacobi function $W = W(s) = ms$ along the curve $s$ such that

$$p_a = \frac{\partial W}{\partial x^a} = m_o c \frac{dx^a}{ds}$$

- Changing the parameter from $s$ to $s'$ gives

$$p_a = m_o c \frac{dx^a}{ds} = m(s') c \frac{dx^a}{ds'} \text{ with } \frac{ds'}{m(s')} \equiv \frac{ds}{m(s)}$$

- Define $d\lambda = \frac{d(s)}{m_o}$ as a universal SHP parameter chosen with respect to some *standard* particle like an electron.

# Generalized Dirac Equation

- Assume that there is no spin angular moment or equivalently that $\tilde{\partial}_s\psi$ is parallel to $\frac{\tilde{ds}}{d\lambda}$ in equation (2) then

$$\tilde{\partial}_s\psi \mid\mid \frac{\tilde{ds}}{d\lambda} \iff \tilde{\partial}_s\psi.\tilde{ds} = d\psi = \frac{d\psi}{d\lambda}d\lambda = \frac{d\psi}{ds}ds$$

- It follows from this and the eigenvector equaton (1) that there is a 1-1 correspondence between a metric and a Dirac wave equation

$$\tilde{ds}\xi = ds\xi \iff \tilde{\partial}_s\psi \equiv \gamma^a\frac{\partial\psi}{\partial x^a} = \frac{\partial\psi}{\partial s}, \qquad (5)$$

- Note that by multiplying both equations of (5) together we have associated with every point of the curve a set of functions $\psi$ such that $\tilde{\partial}_s\psi.\tilde{ds} = \frac{\partial\psi}{\partial s}ds$.

- Note that the usual Dirac equation is a special case of equation (5).

# Equivalent Forms

- The Minkowski (or local tetrad) metric can always be expressed as a null metric in the following way:

$$\begin{aligned} ds^2 &= c^2dt^2 - dx^2 \\ 0 &= c^2dt^2 - c^2d\tau^2 - dx^2 \\ 0 &= c^2dt^2(1-(\frac{d\tau}{dt})^2) - dx^2 \\ 0 &= v^2dt^2 - dx^2 \end{aligned}$$

- This allows us to re-write equation (5) as a Dirac wave equation

$$0 \equiv v\gamma^o dt - \gamma^i dx_i \iff 0 \equiv \gamma^o \frac{1}{v}\frac{\partial\psi}{\partial x^o} - \gamma^a \frac{\partial\psi}{\partial x^a} \tag{6}$$

- Equivalently

$$v^2dt^2 - d\mathbf{x}^2 = 0 \iff \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0$$

# Classical and Quantum Interpretations of a Free Particle

- Consider a particle of rest mass $m_o$ moving along the x-axis in Minkowski space with uniform velocity $v_o$ with respect to proper time and velocity $v$ in the laboratory frame. This means that the uniform motion of the particle with respect to two different frames are related by Lorentz transformations.
- This is encapsulated in HJ function of the particle given by $W = m_o v_o x - m_o c^2 \dot{t} t = -m_o cs$ with $p_o = m_o v_o$, $H = m_o c^2 \dot{t}$ and with corresponding wave equation along a (local) geodesic given by

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \ .$$

# A Classical Particle

- Consider a classical (relativistic) particle with $x = 0$ when $s = 0$, then $x = v_o s$, where $v_o = \frac{dx}{ds}$. In terms of the coordinate system $(x, t)$ of the laboratory frame this can be written as $x = vt$, where $v = (v_o \frac{ds}{dt})$ is constant.
- This information can also be encapsulated in a family of Dirac delta functionals defined by $\psi(W) = \delta_s(W) \equiv W(s)$ which are solutions to the corresponding wave equation.
- The uniform motion implies that both momentum and energy are conserved in all affine frames and consequently $\psi(W)$ is equivalent to

$$\psi((x, t) = \delta(p(x - vt)) = \delta(px - Ht)$$

  in the laboratory frame.
- Here, wave-particle duality emphasizes the particle property. We find the particle in the position $x = vt$ with probability 1

# A Quantum Particle

- The classical particle solution has a physical meaning to the extent that we know the exact initial position and momentum of the particle.
- Things change radically in quantum mechanics because of the Heisenberg's uncertainty principle given by $\Delta x \Delta p \geq \hbar/2$ and $\Delta E \Delta t \geq \hbar/2$ and the fundamental commutator relationships $[x, p] = i\hbar$ and $[t, H] = i\hbar$. It is important to note that that $h$ and $\hbar$ both have the same units as the HJ function $W$.
- Based on the photoelectric effect, we can restrict energy exchanges between particles to be a multiple of nh and write

$$W + \Delta W = W + nh$$

- The following formula follows:

$$\frac{dW}{dt} \approx \frac{\Delta W}{\Delta \tau} = \frac{nh}{\Delta \tau} = nh\nu_o, \qquad \text{where} \qquad \nu_0 = 1/\Delta\tau \ .$$

- Because of the uncertainty principle, the quantum particle characteristics are embedded in the wave function and not vice-versa. This can be especially seen in the zitterbewegung problem.
- In reality the position of a particle constrained to move on the line is unknown because of the uncertainty relations. The best we can do is describe the position by means of a uniform density function $f(x,t) = 1/\xi$ for $x \in [0,\xi]$ and introduce a wave-function on a Hilbert space whose inner product gives the probability distribution.
- This means the HJ equation and Planck's constant $h$ are associated with the wave-fuction $\psi$ and **not** with the probability $f = \langle \psi, \psi \rangle$ .

- Writing the wave function for a free particle ($p^a = constant$) in spinor notation as a self adjoint eigenfunction $\psi(W) = \psi(p^a dx_a)$ and keeeping in mind that for a quantum particle the rest energy can be written as $mc^2 = h\nu$

$$
\begin{aligned}
\psi(W) &= \psi^i\left(\int^x p^a dx_a\right)e_i \\
&= \frac{\exp(iH_o s)}{\sqrt{\xi}}\psi_o \\
&= \frac{\exp(imc^2\tau)}{\sqrt{\xi}}\psi_o \\
&= \frac{\exp(ih\nu_o\tau)}{\sqrt{\xi}}\psi_o
\end{aligned}
$$

- This Zitterbewegung effect suggests that a free quantum particle can be associated with a periodic isotropic vibration.
- In the case of an electron $\nu_o = \frac{mc^2}{\hbar} \approx 10^{21} sec^{-1}$ with compton wavelength $\lambda_c = \frac{\hbar}{mc}$

# Conclusion

Wave-particle duality can be expalined by associating the metrics of relativity with a generalized Dirac equation in the following way:

$$\boxed{ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}dx^a dx^b} \longleftrightarrow \boxed{ds\xi = \gamma_a dx^a \xi}$$

$$\boxed{v^2dt^2 - d\mathbf{x}^2 = 0 \iff \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0}$$

$$\boxed{\frac{\partial^2\psi}{\partial s^2} = \eta_{ab}\frac{\partial^2\psi}{\partial x^a \partial x^b}} \longleftrightarrow \boxed{\frac{\partial\psi}{\partial s} = \gamma^a \frac{\partial\psi}{\partial x^a}}$$

$$\boxed{d\psi = \frac{\partial\psi}{\partial x^a}dx^a}$$

**Note:** The generalized Dirac equation reduces to the usual form of the equation if we let $\psi = Ae^{\kappa W}$, where $W$ is the HJ function derived from the metric, $A$ is an arbitrary constant and $\kappa = \frac{i}{\hbar}$.