# Electric and magnetic fields as explicitly observer dependent four-dimensional vectors and their Lorentz transformations according to Minkowski - Ivezić

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In this paper a geometric approach to the special relativity (SR) is used that is called the "invariant special relativity" (ISR). In the ISR it is considered that in the four-dimensional (4D) spacetime physical laws are geometric, coordinate-free relationships between the 4D geometric, coordinate-free quantities. It is mathematicaly proved that in the ISR the electric and magnetic fields are properly defined vectors on the 4D spacetime. According to the first proof the dimension of a vector field is mathematicaly determined by the dimension of its domain. Since the electric and magnetic fields are defined on the 4D spacetime they are properly defined 4D vectors, the 4D geometric quantities (GQs). As shown in an axiomatic geometric formulation of electromagnetism with only one axiom, the field equation for the bivector field $F$ [T. Ivezić, Found. Phys. Lett. **18,** 401 (2005), arXiv: physics/0412167], the primary quantity for the whole electromagnetism is the bivector field $F$. The electric and magnetic fields 4D vectors $E$ and $B$ are determined in a mathematically correct way in terms of $F$ and the 4D velocity vector $v$ of the observer who measures $E$ and $B$ fields. Furthermore, the proofs are presented that under the mathematicaly correct Lorentz transformations, which are first derived by Minkowski and reinvented and generalized in terms of 4D GQs, e.g., in [T. Ivezić, Phys. Scr. **82,** 055007 (2010)], the electric field 4D vector transforms as any other 4D vector transforms, i.e., again to the electric field 4D vector; there is no mixing with the magnetic field 4D vector $B$, as in the usual transformations (UT) of the 3D fields. Different derivations of these UT of the 3D fields are discussed and objected from the ISR viewpoint. The electromagnetic field of a point charge in uniform motion is considered and it is explicitly shown that 1) the primary quantity is the bivector $F$ and 2) that the observer dependent 4D vectors $E$ and $B$ correctly describe both the electric and magnetic fields for all relatively moving inertial observers and for all bases chosen by them. This formulation with the 4D GQs is in a true agreement, independent of the chosen inertial reference frame and of the chosen system of coordinates in it, with experiments in electromagnetism, e.g., the motional emf. It is shown that the theory with the 4D fields is always in agreement with the principle of relativity, whereas it is not the case with the usual approach with the 3D quantities and their UT.

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## I. INTRODUCTION

Both, in the prerelativistic physics and in Einstein's formulation of special relativity (SR) [1] it is considered that the electric and magnetic fields are the three-dimensional (3D) vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. In the whole physical literature after [1] the usual transformations (UT) of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, the last equations in §6., II. Electrodynamical Part, [1] or, e.g., Eqs. (11.148) and (11.149) in [2], are always considered to be the relativistically correct Lorentz transformations (LT) (boosts) of $\mathbf{E}$ and $\mathbf{B}$. Here, in the whole paper, under the name LT we shall only consider boosts. They are first derived by Lorentz [3] and Poincaré [4] (see also two fundamental Poincaré's papers with notes by Logunov [5]) and independently by Einstein [1]. Then, they are subsequently derived and quoted in almost every textbook and paper on relativistic electrodynamics. According to these UT, the transformed 3D vector $\mathbf{E}'$ is expressed by the mixture of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, Eq. (11.149) in [2], i.e., Eq. (1) here

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E}+\beta\times c\mathbf{B})-(\gamma^2/(1+\gamma))\beta(\beta\cdot\mathbf{E}), \\ \mathbf{B}' &= \gamma(\mathbf{B}-(1/c)\beta\times\mathbf{E})-(\gamma^2/(1+\gamma))\beta(\beta\cdot\mathbf{B}), \end{aligned} \tag{1}$$

where $\mathbf{E}'$, $\mathbf{E}$, $\beta$ and $\mathbf{B}'$, $\mathbf{B}$ are all 3D vectors. It is visible from (1) that, e.g., the electric field $\mathbf{E}$ in one frame is "seen" as slightly changed electric field $\mathbf{E}'$ and an *induced magnetic field* $\mathbf{B}'$ in a relatively moving inertial frame. Henceforward, these UT of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ (1) will be called the Lorentz-Poincaré-Einstein transformations (LPET), according to physicists who discovered them.

In this paper we shall deal with a geometric approach to the SR, which is called the invariant special relativity (ISR). *In the ISR it is considered that in the 4D spacetime physical laws are geometric, coordinate-free relationships between the 4D geometric, coordinate-free quantities.*

These 4D geometric quantities (GQs) are well-defined both theoretically and *experimentally;* they have an independent physical reality. The principle of relativity is automatically satisfied if the physical laws are expressed in terms of 4D GQs. It is not so in the SR [1] in which *the principle of relativity is postulated* outside the mathematical formulation of the theory and it is supposed that it holds for physical laws expressed in terms of 3D quantities. In the ISR physical quantities are represented by the abstract, coordinate-free, 4D GQs. The coordinate-free 4D GQs will be called the abstract quantities (AQs). If some basis in 4D spacetime has been introduced, these AQs are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and *a basis*. Every 4D CBGQ is invariant under the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole CBGQ unchanged. This is the reason for the name ISR. The invariance of a 4D CBGQ under the passive LT reflects the fact that such mathematical, invariant, 4D GQ represents *the same physical quantity* for relatively moving inertial observers and for different bases chosen by them. From the mathematical viewpoint there is no need to introduce CBGQs. However, physicists cannot measure AQs and therefore it is necessary to introduce CBGQs in order to be able to compare the results of experiments with the theory.

In contrast to the ISR the usual SR [1] deals with the Lorentz contraction, the time dilation and the LPET of the 3D vectors **E** and **B**, (1). However, as shown in [6 - 10] and Appendix here, e.g., the Lorentz contraction is ill-defined in the 4D spacetime; it is synchronization dependent and consequently it is not an intrinsic relativistic effect. (Observe that in the second paper in [8], the incorrect quadrupole field of the stationary current loop from the published version is replaced by the dipole field. Therefore, henceforward, if referred to [8] I mean that the corrected version has to be taken into account.) The LT have nothing in common with the Lorentz contraction; the LT cannot connect two *spatial lengths* that are simultaneously determined for relatively moving inertial observers. In the SR the spatial length is defined *as the spatial distance between two spatial points on the (moving) object measured by simultaneity in the rest frame of the observer. The rest length and the Lorentz contracted length are not the same 4D quantity for relatively moving observers and they are not related by the LT, since the transformed length* $L_0(1-\beta^2)^{1/2}$ *is different set of events in the 4D spacetime than the rest length* $L_0$, see in [6] Fig. 3. for the Lorentz contraction and Fig. 4. for the time dilation. Rohrlich [11] named the Lorentz contraction and other transformations which do not refer to the same quantity as the "apparent" transformations (AT), whereas the transformations which refer to the same quantity as the "true" transformations, e.g., the LT. Hence, the other name for the ISR is the "True transformations relativity" ("TT relativity"), which is used, e.g., in [6 - 10]. In the 4D spacetime, as shown in detail in [6 - 10], instead of the Lorentz contraction and the time dilation one has to consider the 4D GQs, the position vectors $x_A$, $x_B$, of the events $A$ and $B$, respectively, the distance vector $l_{AB} = x_B - x_A$ and the spacetime length, which all properly transform under the LT. The essential feature of the geometric approach is that *any abstract 4D geometric quantity (or a 4D CBGQ)*, e.g., the distance vector $l_{AB}$ *is only one quantity, the same quantity in the 4D spacetime* for all relatively moving frames of reference and for all systems of coordinates that are chosen in them, see in [6], Fig. 1. for the spacetime length for a moving rod and Fig. 2. for the spacetime length for a moving clock. In [7] it is explicitly shown that all well-known experiments that test special relativity, e.g., the "muon" experiment, the Michelson-Morley type experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments are in a complete agreement, independently of the chosen synchronization, with the 4D geometric approach, whereas it is not the case with Einstein's approach [1] with the Lorentz contraction and the time dilation if the "radio" ("r") synchronization is used; see two papers in arxiv in [7] in which the "r" synchronization is explicitly used throughout these two papers. Every synchronization is only a convention and physics must not depend on conventions, i.e., no experiment should depend on the chosen synchronization.

Here, in Sec. II, the geometric algebra formalism [12], the standard basis and the $\{r_\mu\}$ basis with the "r" synchronization are briefly discussed. In Secs. IIIA and IIIC it is proved in a mathematically correct way that in the ISR, in the 4D spacetime, the electric and magnetic fields are properly defined vectors on the 4D spacetime, the 4D vectors $E$ and $B$. In the whole text $E$, $B$ will be simply called - vectors - or the 4D vectors, whereas the usual **E**, **B** will be called the 3D vectors. In Secs. IVA and IVB it is proved that from the ISR viewpoint the UT of the 3D fields, i.e., the LPET, are not the mathematically correct LT, because the LT are properly defined on the 4D spacetime and cannot transform the 3D quantities. The LT transform the electric field vector in the same way as any other vector transforms, i.e., again to the electric field vector. Minkowski, in Sec. 11.6 in [13], was the first who introduced the electric and magnetic fields as 4D vectors and derived their correct LT but only with components implicitly taken in the standard basis. This is reinvented and generalized in terms of 4D GQs in [14-20], see also the discussion in [10]. Here, the LT of the 4D vectors $E$ and $B$ will be called the Minkowski-Ivezić LT (MILT). Particularly, in [20], the comparison of our approach with 4D GQs and Minkowski's results is presented in detail. Note, however, that Minkowski never explicitly wrote the LT of the 4D $E$ and $B$, Eqs. (18) - (21) here and he never applied these transformations. Sections IIIA, IIIC, IVA and IVB are the central sections and they contain the most important results that are obtained in this paper. In Secs. VA and VB, for the reader's convenience, the derivations of the UT of the 3D **E** and **B**, the LPET, and the LT of the 4D $E$ and $B$, the MILT, are compared using matrices. In Sec. VII, we discuss the derivation of the LPET from the usual covariant approaches, e.g., from [2]. In Sec. VIII, the

derivation of the LPET from the textbook by Blandford and Thorne (BT) [21] is discussed and objected from the ISR viewpoint. In [21], in contrast to, e.g., [2, 22], a geometric viewpoint is adopted; *the physical laws are stated as geometric, coordinate-free relationships between the geometric, coordinate-free quantities.* Particularly, in Sec. 1.10 in [21], *it is discussed the nature of electric and magnetic fields and they are considered to be the 4D fields.* But, nevertheless, BT also derived the UT of the 3D vectors **E** and **B**, the LPET, their Eq. (1.113), and not the correct LT of the 4D fields, the MILT, Eqs. (18) - (21) here. They have *not* noticed that under the LT the electric field 4D vector must transform as any other 4D vector transforms. In Sec. X, the electromagnetic field of a point charge in uniform motion is investigated and it is explicitly shown that 1) the primary quantity is the bivector $F$ (Eqs. (55) and (56)) and 2) that the observer dependent 4D vectors $E$ and $B$, Eq. (61), correctly describe both the electric and magnetic fields for all relatively moving inertial observers and for all bases chosen by them. In Sec. XI, a brief discussion is presented of the comparison with the experiments on the motional emf. It is shown that the theory with the 4D GQs and their MILT, Eqs. (18) - (21) here, is always in agreement with the principle of relativity, whereas it is not the case with the usual approach with the 3D quantities and their UT, the LPET. In Sec. XII, the discussion of the obtained results is presented and the conclusions are given.

## II. THE GEOMETRIC ALGEBRA FORMALISM. THE $\{r_\mu\}$ BASIS WITH THE "r" SYNCHRONIZATION

Here, as already said, we shall deal either with the abstract, coordinate-free 4D GQs, AQs, or with their representations in some basis, 4D CBGQs comprising *both* components and *a basis*, e.g., the position vector, $x = x^\nu\gamma_\nu$. We shall use the geometric algebra formalism, see, e.g., [12]. The geometric (Clifford) product of two multivectors $A$ and $B$ is written by simply juxtaposing multivectors $AB$. For vectors $a$ and $b$ the geometric product $ab$ decomposes as $ab = a\cdot b + a\wedge b$, where the inner product $a\cdot b$ is $a\cdot b \equiv (1/2)(ab+ba)$ and the outer (or exterior) product $a\wedge b$ is $a\wedge b \equiv (1/2)(ab-ba)$. For the reader's convenience, all equations will be also written with CBGQs in the standard basis. Therefore, the knowledge of the geometric algebra is not required for the understanding of this presentation. The standard basis $\{\gamma_\mu\}$ is a right-handed orthonormal frame of vectors in the Minkowski spacetime $M^4$ with $\gamma_0$ in the forward light cone, $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ $(k = 1, 2, 3)$. The $\gamma_\mu$ generate by multiplication a complete basis for the spacetime algebra: 1, $\gamma_\mu$, $\gamma_\mu\wedge\gamma_\nu$, $\gamma_\mu\gamma_5$, $\gamma_5$ ($2^4 = 16$ independent elements). $\gamma_5$ is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. The $\{\gamma_\mu\}$ basis corresponds to Einstein's system of coordinates in which the Einstein synchronization of distant clocks [1] and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference. Here, we shall also introduce another basis, the $\{r_\mu\}$ basis with the "r" synchronization, see also [6, 7, 23]. The "r" synchronization is commonly used in everyday life, [24]. If the observers who are at different distances from the studio clock set their clocks by the announcement from the studio then they have synchronized their clocks with the studio clock according to the "r" synchronization.

The unit vectors in the $\{\gamma_\mu\}$ basis and the $\{r_\mu\}$ basis are connected as $r_0 = \gamma_0$, $r_i = \gamma_0+\gamma_i$. Hence, the components $g_{\mu\nu,r}$ of the metric tensor are $g_{ii,r} = 0$, and all other components are $= 1$. Obviously it is completely different than in the $\{\gamma_\mu\}$ basis, i.e. than the Minkowski metric, which, here, is chosen to be $g_{\mu\nu} = diag(1,-1,-1,-1)$. (Note that in [6 - 9] the Minkowski metric is $g_{\mu\nu} = diag(-1,1,1,1)$.) Then, according to Eq. (4) from [6], one can use $g_{\mu\nu,r}$ to find the transformation matrix $R^\mu_{\ \nu}$ that connects the components in the $\{\gamma_\mu\}$ and the $\{r_\mu\}$ bases. The only components that are different from zero are

$$R^\mu_{\ \mu} = -R^0_{\ i} = 1. \tag{2}$$

The inverse matrix $(R^\mu_{\ \nu})^{-1}$ connects the "old" basis, $\{\gamma_\mu\}$, with the "new" one, $\{r_\mu\}$. The components of any vector are connected in the same way as the components of the position vector $x$ are connected, i.e., as

$$x^0_r = x^0 - x^1 - x^2 - x^3, \quad x^i_r = x^i. \tag{3}$$

This reveals that *in the* $\{r_\mu\}$ *basis the space* **r** *and the time* $t$ *cannot be separated*; the "3+1 split" of the spacetime into space + time is *impossible*. Note that there is the zeroth component of $x$ in the $\{r_\mu\}$ basis, $x^0_r \neq 0$, even if in the standard basis $x^0 = 0$, but the spatial components $x^i \neq 0$. This means that *in the 4D spacetime only the position vector* $x$, $x = x^\mu\gamma_\mu = x^\mu_r r_\mu$, *is properly defined quantity.* In general, the position in the 3D space **r** and the time $t$ have not an independent reality in the 4D spacetime. Although the Einstein and the "r" synchronizations are completely different they are equally well physical and relativistically correct synchronizations. *Every synchronization is only a convention and physics must not depend on conventions.* An important consequence of the result that in the 4D spacetime **r** and $t$ are not well-defined is presented in Sec. 4 in [23]. There, it is shown that only the world parity $W$, $Wx = -x$, is well defined in the 4D spacetime and not the usual $T$ and $P$ inversions. We remark that in order

to treat different bases on an equal footing the general transformation matrix $T^{\mu}_{\nu}$ is presented in [6], Eq. (4), that connects the $\{\gamma_\mu\}$ basis and some other basis, e.g., the $\{r_\mu\}$ basis, in the same reference frame. That matrix $T^{\mu}_{\nu}$ is expressed in terms of the basis components of the metric tensor and for the connection with the $\{r_\mu\}$ basis it is given by Eq. (2). It is worth mentioning that in Eq. (1) in [6] it is derived such form of the LT, which is independent of the chosen system of coordinates, including different synchronizations.

## III. IN THE ISR THE ELECTRIC AND MAGNETIC FIELDS ARE WELL DEFINED 4D VECTORS

### A. Oziewicz's proof

There is a simple but very strong and completely correct mathematical argument, which is stated by Oziewicz, e.g., in [25]. There, it is explained that an individual vector has no dimension; the dimension is associated with the vector space and with the manifold where this vector is tangent. Hence, *what is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e., the dimension of its domain. In general, the dimension of a vector field that is defined on a n-dimensional space is equal - n. The electric and magnetic fields are defined on a 4D space, i.e., the spacetime. They are always functions of the position vector $x$. This means that they are not the usual 3D fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, but they are properly defined vectors on the 4D spacetime, $E(x)$ and $B(x)$. This fact determines that such vector fields, when represented in some basis, have to have four components (some of them can be zero).* This is a fundamental argument and it cannot be disputed in any way. It is very surprising that this argument is not used in physics much earlier. For an exact mathematical proof of that argument see, e.g., Chapter I in an undergraduate text in mathematics (the first book in [26]) or Chapter II in an advanced text (the second book in [26]).

The mentioned argument holds in the same measure for the polarization vector $P(x)$ and the magnetization vector $M(x)$, which are discussed in detail in [27, 28, 10]. In [27] the electromagnetic field equations for moving media are presented, whereas in [28] the constitutive relations and the magnetoelectric effect for moving media are investigated from the geometric point of view. In addition to Oziewicz's proof we note that in the 4D spacetime we *always* have to deal with correctly defined vectors $E(x)$, $B(x)$, $P(x)$, $M(x)$, etc. even in the usual static case, i.e., if the usual 3D fields $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, .. do not explicitly depend on the time $t$. The reason is that if *in the 4D spacetime* the standard basis is used then the LT cannot transform the spatial coordinates from one frame only to spatial coordinates in a relatively moving inertial frame of reference. What is static case for one inertial observer is not more static case for relatively moving inertial observer, but a time dependent case. Furthermore, if an observer uses the "r" synchronization and not Einstein's synchronization, then, as seen from (3), the space and time are not separated and the usual 3D vector $\mathbf{r}$ is meaningless. If the principle of relativity has to be satisfied and the physics must be the same for all inertial observers and for $\{\gamma_\mu\}$, $\{r_\mu\}$, $\{\gamma'_\mu\}$, etc. bases which they use, then the properly defined quantity is the position vector $x$,

$$x = x^{\nu}\gamma_\nu = x'^{\nu}\gamma'_\nu = x^{\nu}_{r} r_\nu = x'^{\nu}_{r} r'_\nu, \tag{4}$$

and not $\mathbf{r}$ and $t$. In (4), the primed quantities in both bases $\{\gamma_\mu\}$ and $\{r_\mu\}$ are the Lorentz transforms of the unprimed ones. For the $\{r_\mu\}$ basis and the LT in that basis see [6]. Consequently, in the 4D spacetime, e.g., the electric field is properly defined as the vector $E(x)$ for which, in the same way as in (4), the relation (26) given below holds.

### B. Briefly about the $F$ formulation

In this section for the sake of completeness and for better understanding of the whole exposition we briefly repeat main results from [29]. In [29] an axiomatic geometric formulation of electromagnetism with only one axiom, the field equation for the bivector field $F$, Eq. (4) in [29], is constructed. There, it is shown that the bivector $F = F(x)$, which represent the electromagnetic field, can be taken as the primary quantity for the whole electromagnetism. It yields a complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. If the field equation for $F$ is written with AQs it becomes

$$\partial \cdot F + \partial \wedge F = j/\varepsilon_0 c, \tag{5}$$

where the source of the field is the charge-current density vector $j(x)$, Eq. (4) in [29]. If $j(x)$ is the sole source of $F$ then the general solution for $F$ with AQs is given by Eq. (8) in [29]. Particularly, the general expression for $F$ for an arbitrary motion of a charge is given by Eq. (10) in [29] with AQs and as a CBGQ in the $\{\gamma_\mu\}$ basis by Eq. (11) in [29]. $F$ of a point charge in uniform motion as an AQ is given by Eq. (12) in [29], i.e., Eq. (55) here. The

components in the standard basis $F^{\alpha\beta}$ from that Eq. (11) in [29] are the same as the usual result from Chapter 14 in [2]. Note that in Sec. 2.4. in [29] the integral form of Eq. (5) is also presented and discussed. If the equation for $F$ (5) is written with CBGQs in the $\{\gamma_\mu\}$ basis it becomes Eq. (5) in [29],

$$\partial_\alpha F^{\alpha\beta}\gamma_\beta - \partial_\alpha \, {}^*F^{\alpha\beta}\gamma_5\gamma_\beta = (1/\varepsilon_0 c)j^\beta\gamma_\beta, \tag{6}$$

where the usual dual tensor (components) is ${}^*F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. From that equation one easily finds the usual covariant form (only the basis components of the 4D GQs in the $\{\gamma_\mu\}$ basis) of the field equations as Eq. (6) in [29],

$$\partial_\alpha F^{a\beta} = j^\beta/\varepsilon_0 c, \quad \partial_\alpha \, {}^*F^{\alpha\beta} = 0. \tag{7}$$

These two equations for *the components in the standard basis* $F^{\alpha\beta}$ are the equations (11.141) and (11.142) in [2].

In the same paper, [29], it is also shown that this formulation with the $F$ field is in a complete agreement with the Trouton-Noble experiment, i.e., in the approach with $F$ as a 4D GQ there is no Trouton-Noble paradox. It is clearly visible from [29] and this short presentation that, in principle, *the components* $F^{\alpha\beta}$ of the electromagnetic field tensor, i.e., of the bivector $F$ here and in [29], *have nothing to do with the components of the 3D vectors* $\mathbf{E}$ *and* $\mathbf{B}$. *The whole* $F$ *is a physically measurable quantity by the Lorentz force density,* $K_{(j)} = F \cdot j/c$, Eq. (27) in [29], *or, for a charge* $q$ *by the Lorentz force*

$$K_L = (q/c)F \cdot u, \tag{8}$$

where $u$ is the 4D velocity vector of a charge $q$ (it is defined to be the tangent to its world line).

It is worth noting that the expression for the Lorentz force density, $K_{(j)} = F \cdot j/c$, is directly derived from the field equation for $F$ (5). Similarly, in [29], the coordinate-free expressions for the stress-energy vector $T(n)$ (Eqs. (37) and (38)) and the quantities derived from $T(n)$, the energy density $U$ (scalar, Eq. (39)), the Poynting 4D vector $S$ (Eq. (40)), the momentum density 4D vector $g = (1/c^2)S$ (Eq. (42)), the angular momentum density $M$, $M = (1/c)T(n) \wedge x$ (bivector, Eq. (43)), the local charge conservation law (Eq. (48)) and the local energy-momentum conservation law (Eqs. (49) and (50)) are all directly derived from that field equation (5). In that axiomatic geometric formulation from [29] $T(n)$ is the most important quantity for the momentum and energy of the electromagnetic field. $T(n)$ is a vector-valued linear function on the tangent space at each spacetime point $x$ describing the flow of energy-momentum through a hypersurface with normal $n = n(x)$, $T(n) = -(\varepsilon_0/2)\left[(F \cdot F)n + 2(F \cdot n) \cdot F\right]$, Eq. (37) in [29]. In Eq. (38) in [29] $T(n)$ is written in a new form as a sum of $n$-parallel part ($n-\parallel$) and $n$-orthogonal part ($n-\perp$), $T(n) = -(\varepsilon_0/2)\left[(F \cdot F) + 2(F \cdot n)^2\right] n - \varepsilon_0\left[(F \cdot n) \cdot F - (F \cdot n)^2 n\right]$. The first term is $n-\parallel$ part and it yields the energy density $U$, $U = n \cdot T(n)$, $U = -(\varepsilon_0/2)\left[(F \cdot F) + 2(F \cdot n)^2\right]$, whereas the second term is $n-\perp$ part and it is $(1/c)S$, where $S$ is the Poynting 4D vector, $S = -\varepsilon_0 c\left[(F \cdot n) \cdot F - (F \cdot n)^2 n\right]$. and, as can be seen, $n \cdot S = 0$. Thus $T(n)$ is expressed by $U$ and $S$ as in Eq. (41) in [29], $T(n) = Un + (1/c)S$;

$$\begin{aligned} T(n) &= Un + (1/c)S, \quad U = -(\varepsilon_0/2)\left[(F \cdot F) + 2(F \cdot n)^2\right], \\ S &= -\varepsilon_0 c\left[(F \cdot n) \cdot F - (F \cdot n)^2 n\right]. \end{aligned} \tag{9}$$

Observe that $T(n)$ as a whole quantity, i.e., the combination of $U$ and $S$ enters into a fundamental physical law, the local energy-momentum conservation law, $\partial \cdot T(n) = 0$, for the free fields, Eq. (49) in [29]. This means, as stated in [29], that only $T(n)$, as a whole quantity, does have a physically correct interpretation. In [29] this viewpoint is nicely illustrated considering *an apparent paradox in the usual 3D formulation* in which the 3D Poynting vector $S$ is interpreted as an energy flux due to the propagation of the 3D fields. If such an interpretation of $S$ is adopted then there is a paradox for the case of an uniformly accelerated charge, e.g., Sec. 6.8 in [2]. In that case, the 3D $S$ is $= 0$ (there is no energy flow) but at the same time the 3D $U$ is $\neq 0$ (there is an energy density) for the field points on the axis of motion. The obvious question is how the fields propagate along the axis of motion to give that $U \neq 0$. In the formulation with 4D GQs the important quantity is $T(n)$ and not $S$ and $U$ taken separately. $T(n)$ is $\neq 0$ everywhere on the axis of motion and *the local energy-momentum conservation law* with $T(n)$ *holds everywhere.*

### C. Proof by the use of the decomposition of $F$

In contrast to the usual covariant approach, which deals with the identification of components, see Sec. VII, Eqs (41) and (43), it is possible to construct in a mathematically correct way the 4D vectors of the electric and magnetic fields using the decomposition of $F$. There is *a mathematical theorem according to which any antisymmetric tensor*

*of the second rank can be decomposed into two space-like vectors and the unit time-like vector.* For the proof of that theorem in geometric terms see, e.g., [30].

If that theorem is applied to the bivector $F$ then it is obtained that

$$F = E \wedge v/c + (IcB) \cdot v/c, \tag{10}$$

where the electric and magnetic fields are represented by vectors $E(x)$ and $B(x)$, see, e.g., [14 - 16, 29]. The unit pseudoscalar $I$ is defined algebraically without introducing any reference frame as in [12] (Hestenes and Sobczyk). If $I$ is represented in the $\{\gamma_\mu\}$ basis it becomes $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_5$. The vector $v$ in the decomposition (10) is interpreted as the velocity vector of the observer who measures $E$ and $B$ fields. Then $E(x)$ and $B(x)$ are defined with respect to $v$, i.e., with respect to the observer, as

$$E = F \cdot v/c, \quad B = -(1/c)I(F \wedge v/c). \tag{11}$$

It also holds that $E \cdot v = B \cdot v = 0$; both $E$ and $B$ are space-like vectors. If the decomposition (10) is written with the CBGQs in the $\{\gamma_\mu\}$ basis it becomes

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \; F^{\mu\nu} = (1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta}v_\alpha B_\beta, \tag{12}$$

where $\gamma_\mu \wedge \gamma_\nu$ is the bivector basis. If the equations for $E$ and $B$ (11) are written with the CBGQs in the $\{\gamma_\mu\}$ basis they become

$$E = E^\mu\gamma_\mu = (1/c)F^{\mu\nu}v_\nu\gamma_\mu, \quad B = B^\mu\gamma_\mu = (1/2c^2)\varepsilon^{\mu\nu\alpha\beta}F_{\nu\alpha}v_\beta\gamma_\mu. \tag{13}$$

All these relations, (10) - (13) are the mathematically correct definitions. They are first reported (only components implicitly taken in the standard basis) by Minkowski in Sec. 11.6 in [13], see also [20].

Let us introduce the $\gamma_0$ - frame; the frame of "fiducial" observers for which $v = c\gamma_0$ and in which the standard basis is chosen. Therefore, in the $\gamma_0$ - frame, e.g., $E$ becomes $E = F \cdot \gamma_0$. It can be shown that in the $\gamma_0$ - frame $E \cdot \gamma_0 = B \cdot \gamma_0 = 0$, which means that $E$ and $B$ are orthogonal to $\gamma_0$; they refer to the 3D subspace orthogonal to the specific timelike direction $\gamma_0$. If $E$ and $B$ are written as CBGQs in the standard basis they become

$$\begin{aligned} E &= E^\mu\gamma_\mu = 0\gamma_0 + F^{i0}\gamma_i, \\ B &= B^\mu\gamma_\mu = 0\gamma_0 + (1/2c)\varepsilon^{0ijk}F_{kj}\gamma_i. \end{aligned} \tag{14}$$

Note that $\gamma_0 = (\gamma_0)^\mu\gamma_\mu$ with $(\gamma_0)^\mu = (1, 0, 0, 0)$. Hence, in the $\gamma_0$ - frame the temporal components of $E$ and $B$ are zero and only the spatial components remain

$$E^0 = B^0 = 0, \quad E^i = F^{i0}, \quad B^i = (1/2c)\varepsilon^{0ijk}F_{kj}. \tag{15}$$

It is visible from (14) and (15) that $E^i$ and $B^i$ are the same as the components of the 3D $\mathbf{E}$ and $\mathbf{B}$, Sec. VII, Eq. (41), i.e., the same as in Eq. (11.137) in [2]. However, there are very important differences between the identifications (41) and Eqs. (14) and (15). The components of $\mathbf{E}$ and $\mathbf{B}$ in (41) are not the spatial components of the 4D quantities. They transform according to the LPET, Sec. VII, Eq. (45). The antisymmetric $\varepsilon$ tensor in (41) and (43) is a third-rank antisymmetric tensor. On the other hand, the components of $E$ and $B$ in (14) and (15) are the spatial components of the 4D geometric quantities that are taken in the standard basis. They transform according to the MILT that are given below, Eq. (19). The antisymmetric $\varepsilon$ tensor in (14) and (15) is a fourth-rank antisymmetric tensor. Furthermore, as shown in Sec. VII, Eqs. (46) and (47), the identifications (41) and (43) do not hold in the $\{r_\mu\}$ basis. But, the relations (13) hold for any chosen basis, including the $\{r_\mu\}$ basis, e.g.,

$$E = E^\nu\gamma_\nu = E^\nu_r r_\nu = (1/c)F^{\mu\nu}_r v_{\nu,r} r_\mu. \tag{16}$$

This can be easily checked using the above mentioned matrix $R^\mu_\nu$. Thus, for the components of vector $E$ it also holds that

$$E^0_r = E^0 - E^1 - E^2 - E^3, \quad E^i_r = E^i. \tag{17}$$

From these relations it follows that there is the zeroth component of $E$ in the $\{r_\mu\}$ basis, $E^0_r \neq 0$, even if it is $= 0$ in the standard basis, $E^0 = 0$, but the spatial components $E^i \neq 0$. This again shows that *the components taken alone*

*are not physical*. The whole consideration presented here explicitly reveals that in the 4D spacetime, from the ISR viewpoint, the usual identifications (41) and (43) are not mathematically correct and that in the ISR

*the electric field $E$ is a vector (4D vector); it is an inner product of a bivector $F$ and the velocity vector $v$ of the observer who measures fields.*

It is worth mentioning that in the 4D spacetime the mathematically correct relations (10) - (13) are already firmly theoretically founded and they are known to many physicists. The recent example is, e.g., in [31]; it is only the electric part (the magnetic part is zero there). Similarly, in the component form these relations are presented, e.g., in [32] and in the basis-free form with AQs in [33, 21, 25, 30]. But, it has to be noted that from all of them only Oziewicz, see [25] and references to his papers in it, *exclusively* deals with the abstract, basis-free 4D quantities. He correctly considers from the outset that in the 4D spacetime such quantities are physical quantities and not the usual 3D quantities. All others, starting with Minkowski [13], are not consistent in the use of the 4D electric and magnetic fields. They use the usual 3D fields **E** and **B** together with the 4D fields considering that the 3D fields are physically measurable quantities and that their LPET are the mathematically correct LT, whereas the 4D fields are considered to be only mathematical, auxiliary, quantities. Minkowski [13] introduced only in Sec. 11.6 the 4D fields and their LT. In other sections he also dealt with the 3D fields and their LPET. This is explained in detail in [20], which is under the title: "Lorentz transformations of the electric and magnetic fields according to Minkowski."

## IV. THE PROOFS THAT IN THE ISR THE ELECTRIC FIELD VECTOR TRANSFORMS AGAIN TO THE ELECTRIC FIELD VECTOR

In the ISR, as proved in Secs. IIIA and IIIC, the electric field is properly defined vector on the 4D spacetime and the same holds for the magnetic field. Hence, *under the LT*, e.g., *the electric field vector must transform as any other vector transforms*, i.e., *again to the electric field vector*; *there is no mixing with the magnetic field vector $B$*. In [17] the same result is obtained for the electric field as a bivector and for the magnetic field as well. This will be explicitly shown both for the active LT in IVA and for the passive LT in IVB.

### A. Proof with the coordinate-free quantities, AQs, and the active LT

Regarding the correct LT of $E$ and $B$, i.e., MILT, let us start from the definition with the coordinate-free quantities $E = c^{-1}F \cdot v$ and with *the active LT. Mathematically,* as noticed by Oziewicz [25], *an active LT must act on all tensor fields from which the vector field $E$ is composed, including an observer's time-like vector field*. This means that the mathematically correct active LT of $E = c^{-1}F \cdot v$ are $E' = c^{-1}F' \cdot v'$; both $F$ and $v$ are transformed. It was first discovered by Minkowski in Sec. 11.6 in [13] but with components implicitly taken in the standard basis and, as already said, reinvented and generalized in terms of 4D GQs in [14-20], see also Secs. 5 and 6 in [10]. Since this issue is discussed at great length in [20] and again in [10] we confine our remarks here to a summary of the conclusions reached in [20]. As explicitly shown, e.g., in [20], in the geometric algebra formalism *any multivector $N$ transforms by the active LT in the same way,* i.e., as $N \to N' = RN\widetilde{R}$, where $R$ is given by Eq. (10) in [20] (Eq. (39) in [10]); for boosts in an arbitrary direction the rotor $R$ is $R = (1 + \gamma + \gamma\gamma_0\beta)/(2(1+\gamma))^{1/2}$, where $\gamma = (1-\beta^2)^{-1/2}$, the vector $\beta$ is $\beta = \beta s$, $\beta$ on the r.h.s. of that equation is the scalar velocity in units of $c$ and $s$ is not the basis vector but any unit space-like vector orthogonal to $\gamma_0$. The reverse $\widetilde{R}$ is defined by the operation of reversion according to which $\widetilde{AB} = \widetilde{B}\widetilde{A}$, for any multivectors $A$ and $B$, see Sec. 3 in [20] (Sec. 5 in [10]). Hence, the vector $E = c^{-1}F \cdot v$ transforms by *the mathematically correct* active LT $R$ into $E' = RE\widetilde{R} = c^{-1}R(F \cdot v)\widetilde{R} = c^{-1}(RF\widetilde{R}) \cdot (Rv\widetilde{R}) = c^{-1}F'v'$. If $v = c\gamma_0$ is taken in the expression for $E$ then $E$ becomes $E = F \cdot \gamma_0$ and it transforms according to MILT as in Eq. (12) in [20], i.e., that *both $F$ and $\gamma_0$* are transformed by the LT, $E = F \cdot \gamma_0 \longrightarrow E' = R(F \cdot \gamma_0)\widetilde{R} = (RF\widetilde{R}) \cdot (R\gamma_0\widetilde{R})$. Hence, the explicit form for $E'$ with the abstract, coordinate-free quantities is given by Eq. (13) in [20],

$$E' = E + \gamma(E \cdot \beta)\{\gamma_0 - (\gamma/(1+\gamma))\beta\}. \tag{18}$$

In (18) $\beta$ is a vector. That equation is first reported in [20]. In the ISR Eq. (18) with the 4D vectors replaces Eq. (1) with the 3D vectors; both equations are with geometric quantities but the 3D vectors from (1) are well defined in the 3D space whereas the 4D vectors from (18) are well defined in the 4D spacetime. In the standard basis and for boosts in the direction $x^1$ the components of that $E'$ are

$$E'^{\mu} = (E'^0 = -\beta\gamma E^1,\ E'^1 = \gamma E^1,\ E'^{2,3} = E^{2,3}). \tag{19}$$

Under the active LT the electric field vector $E = F \cdot \gamma_0$ (as a CBGQ it is $E = E^{\mu}\gamma_{\mu} = 0\gamma_0 + F^{i0}\gamma_i$) is transformed into a new electric field vector $E'$, (18). Note that under the active LT the components are changed, (19), but the basis remains unchanged,

$$E'^{\nu}\gamma_{\nu} = -\beta\gamma E^1\gamma_0 + \gamma E^1\gamma_1 + E^2\gamma_2 + E^3\gamma_3, \tag{20}$$

Eq. (14) in [20] (Eq. (43) in [10]). *The components $E^\mu$ transform by the LT again to the components $E'^\mu$ and there is no mixing with $B^\mu$.* In general, the LT of the components $E^\mu$ (in the $\{\gamma_\mu\}$ basis) of $E = E^\mu\gamma_\mu$ are given as

$$E'^0 = \gamma(E^0 - \beta E^1),\ E'^1 = \gamma(E^1 - \beta E^0),\ E'^{2,3} = E^{2,3}, \tag{21}$$

for a boost along the $x^1$ axis, i.e., the same LT as for any other 4D vector.

On the other hand, if in $E = F\cdot\gamma_0$ only $F$ is transformed by the active LT and not $\gamma_0$, which is not a mathematically correct procedure, then the components of that $E'_F$ will be denoted as $E'^\mu_F$ and they are

$$E'^\mu_F = (E'^0_F = 0, E'^1_F = E^1, E'^2_F = \gamma(E^2 - c\beta B^3), E'^3_F = \gamma(E^3 + c\beta B^2)), \tag{22}$$

Eq. (17) in [20] (Eq. (46) in [10]), i.e., (27) below. *The transformations of the spatial components (taken in the standard basis) of E are exactly the same as the transformations of $E_{x,y,z}$ from Eq.* (11.148) *in* [2], i.e., as in Eq. (45) here. However, from $E = F\cdot\gamma_0$ it follows that the components of $E$ are $E^\mu = (E^0 = 0,\ E^1,\ E^2,\ E^3)$. Hence, *if only $F$ is transformed by the LT then the temporal components of both $E$ and $E'_F$ are zero*, $E^0 = E'^0_F = 0$, which explicitly reveals that from the ISR viewpoint such transformations are not the mathematically correct LT; the LT cannot transform $E^0 = 0$ again to $E'^0_F = 0$. This proves that from the ISR viewpoint the transformations (19) in which *both $F$ and $\gamma_0$* are transformed are the mathematically correct LT, the MILT.

### B. Proof with CBGQs and the passive LT

If $E$ is written as a CBGQ, i.e., as in (13), then we have to use the passive LT. This is discussed at length in, e.g., [14-19], [10], but for the sake of completeness and for better understanding we repeat the short proof from [19]. For example, in the $\gamma_0$ - frame $E$ is given as

$$E = E^\mu\gamma_\mu = [(1/c)F^{i0}v_0]\gamma_i = 0\gamma_0 + E^i\gamma_i \tag{23}$$

For boosts in the $\gamma_1$ direction and if *both $F^{i0}$ and $v_0$* are transformed by the LT, i.e., the MILT, then, as for any other CBGQ, it holds that

$$E = E^\mu\gamma_\mu = [(1/c)F'^{\mu\nu}v'_\nu]\gamma'_\mu = E'^\mu\gamma'_\mu, \tag{24}$$

where, again, the components $E'^\mu$ are the same as in (19), see [19]. On the other hand, if only $F^{i0}$ is transformed but not $v_0$ then the transformed components $E'^\mu_F$ are again the same as in (45) and the same objections as in Sec. IV.A hold also here. In addition, it can be easily checked that

$$E'^\mu_F\gamma'_\mu \neq E^\mu\gamma_\mu, \tag{25}$$

which additionaly proves that from the ISR viewpoint the transformations in which only $F$ is transformed are not the mathematically correct LT and accordingly the same holds for the transformations given by Eqs. (11.148) ((22) here) and (11.149) from [2]. They are the LPET of the components in the standard basis of the 3D vectors **E** and **B** that do not refer to the same 4D quantity. On the other hand, as can be seen from the above discussion, if $E$ is written as a CBGQ then, as for any other 4D CBGQ, e.g., as in (4), it holds that

$$E = E^\nu\gamma_\nu = E'^\nu\gamma'_\nu = E^\nu_r r_\nu = E'^\nu_r r'_\nu. \tag{26}$$

Again, as in (4), the primed quantities in both bases $\{\gamma_\mu\}$ and $\{r_\mu\}$ are the Lorentz transforms of the unprimed ones.

## V. THE COMPARISON OF THE DERIVATIONS OF THE LPET AND THE MILT USING MATRICES (THE COMPONENTS IN THE STANDARD BASIS)

### A. The electric and magnetic fields as vectors

For the reader's convenience the same results as in Secs. III - III.C can be obtained explicitly using the matrices. We write the relation $E^\mu = c^{-1}F^{\mu\nu}v_\nu$ in the $\gamma_0$ - frame, i.e., for $v = c\gamma_0$. From the matrix for $F^{\mu\nu}$ and $v_\nu = (c, 0, 0, 0)$ one finds $E^\mu = (0, F^{10} = E^1, F^{20} = E^2, F^{30} = E^3)$.

Then, *for the LPET only $F^{\mu\nu}$ is transformed by the LT but not the velocity of the observer $v = c\gamma_0$.* The Lorentz transformed $F^{\mu\nu}$ is (symbolically) $F' = AF\widetilde{A}$; here $A$, $F$, .. denote matrices. This relation can be written with

components as $F'^{\mu\nu} = A^\mu_\rho F^{\rho\sigma}\widetilde{A}^\nu_\sigma$. The matrix $A$ is the boost in the direction $x^1$ (in the standard basis) and it is written in Eq. (33). $A$ is also given by Eq. (11.98) in [2] (with only $\beta_1 \neq 0$) and $\widetilde{A}$ is obtained transposing $A$. The transformed components $E'^\mu_F$ are obtained as $E'^\mu_F = c^{-1}F'^{\mu\nu}v_\nu$, or explicitly with matrices as

$$\begin{bmatrix} 0 & -F'^{10} & -F'^{20} & -F'^{30} \\ E^1 & 0 & -F'^{21} & -F'^{31} \\ \gamma(E^2-\beta cB^3) & \gamma(-\beta E^2+cB^3) & 0 & -F'^{32} \\ \gamma(E^3+\beta cB^2) & \gamma(-\beta E^3-cB^2) & cB^1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ E^1 \\ \gamma(E^2-\beta cB^3) \\ \gamma(E^3+\beta cB^2) \end{bmatrix}, \tag{27}$$

where the first matrix is the Lorentz transformed $F^{\mu\nu}$, i.e., $F'^{\mu\nu}$, and the second matrix is $c^{-1}v^\mu = \gamma^\mu_0$. The components $E'^\mu_F$ are already written in Eq. (22). As seen from (27) the transformed zeroth component $E'^0_F$ is again $= 0$, which shows, as previously stated, that from the ISR viewpoint, such transformations cannot be the mathematically correct LT; the LT cannot transform the 4D vector with $E^0 = 0$ into the 4D vector with $E'^0_F = 0$. Furthermore, it can be simply checked using (27) that for the CBGQs holds

$$E'^\mu_F\gamma'_\mu \neq E^\mu\gamma_\mu, \tag{28}$$

where $E'^\mu_F$ is from (27). This is the same as in (25), i.e., it additionally proves that $E'^\mu_F$ is not obtained by the mathematically correct LT from $E^\mu$.

*Under the mathematically correct LT, the MILT, both $F^{\mu\nu}$ and the velocity of the observer $v = c\gamma_0$ are transformed.* Then (symbolically)

$$E = c^{-1}F\cdot v \longrightarrow E' = c^{-1}F'\cdot v' = c^{-1}(AF\widetilde{A})(A^{-1}v) = A(c^{-1}Fv) = AE, \tag{29}$$

where, here, $E$, $F$, $v$, $A$, $F'$, ... denote matrices. Hence, $E'^\mu$ can be written as

$$E'^\mu = c^{-1}F'^{\mu\nu}v'_\nu = c^{-1}(A^\mu_\rho F^{\rho\sigma}\widetilde{A}^\nu_\sigma)((A^{-1})^\alpha_\nu v_\alpha) = A^\mu_\rho(c^{-1}F^{\rho\alpha}v_\alpha). \tag{30}$$

Using the explicit matrices $c^{-1}A^{-1}v$ is given as

$$c^{-1}A^{-1}v = c^{-1}\begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma \\ \beta\gamma \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

and $E'^\mu$ is $E'^\mu = c^{-1}F'^{\mu\nu}v'_\nu$, i.e.,

$$\begin{bmatrix} 0 & -E^1 & -F^{2'0'} & -F^{3'0'} \\ E^1 & 0 & -F^{2'1'} & -F^{3'1'} \\ \gamma(E^2-\beta cB^3) & \gamma(-\beta E^2+cB^3) & 0 & -F^{3'2'} \\ \gamma(E^3+\beta cB^2) & \gamma(-\beta E^3-cB^2) & cB^1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \gamma \\ \beta\gamma \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\beta\gamma E^1 \\ \gamma E^1 \\ E^2 \\ E^3 \end{bmatrix}, \tag{32}$$

where again the first matrix is $F'^{\mu\nu}$, as in (27), but the second matrix is the Lorentz transformed 4D velocity of the observer, i.e., it is given by Eq. (31). Observe that the same result for $E'^\mu$ is obtained from $E'^\mu = A^\mu_\nu E^\nu$,

$$E'^\mu = A^\mu_\nu E^\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix} = \begin{bmatrix} -\beta\gamma E^1 \\ \gamma E^1 \\ E^2 \\ E^3 \end{bmatrix}. \tag{33}$$

The components $E'^\mu$ are the same as in (19). This result clearly shows that from the ISR viewpoint the transformations in which *both* $F$ and the velocity of the observer $v$ are transformed are the mathematically correct LT, the MILT; *under such LT, i.e., MILT, the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms.*

As an additional proof of that result it can be simply checked using (33) that for the CBGQs $E^\nu\gamma_\nu$, $E'^\nu\gamma'_\nu$, ... again holds the relation (26), $E = E^\nu\gamma_\nu = E'^\nu\gamma'_\nu = E^\nu_r r_\nu = E'^\nu_r r'_\nu$, as for any other CBGQ.

### B. The electric and magnetic fields as bivectors

In [17] the same result about the fundamental difference between the LPET and the correct LT, MILT, is obtained representing the electric and magnetic fields by bivectors. The representation by bivectors is used, e.g., in [12] and they derived the UT in which the components of the transformed electric field bivector are expressed by the combination of components of the electric and magnetic field bivectors like in (45). In the $\gamma_0$ - frame the electric field bivector $\mathbf{E}_H$ is determined from the electromagnetic field bivector, Eq. (2) in [17], $\mathbf{E}_H = (F\cdot\gamma_0)\gamma_0 = (1/2)(F-\gamma_0 F\gamma_0)$. In Sec. 5 in [17] the derivation of the LPET from [12] is presented. The space-time split is made and accordingly the space-space components are zero for the matrix of the electric field bivector $(\mathbf{E}_H)^{\mu\nu}$, Eq. (5) in [17], i.e., $(\mathbf{E}_H)^{i0} = F^{i0} = E^i$, $(\mathbf{E}_H)^{ij} = 0$. Then, in [12], the same is supposed to hold for the electric field bivector that is transformed by the LPET, Eqs. (18) and (19) in [17]. The transformed electric field bivector $\mathbf{E}'_{H,at}$ is not obtained in the way in which all other multivectors transform, but it is obtained that only F is transformed whereas $\gamma_0$ is not transformed, Eq. (16) in [17], $\mathbf{E}'_{H,at} = (1/2)[F'-\gamma_0 F'\gamma_0] = (F'\cdot\gamma_0)\gamma_0$. This is the treatment from [12]. They have not noticed that such transformations cannot be the correct LT because the LT cannot transform the matrix (5) in [17] in which the space-space components are zero to the matrix (18) in [17] in which again the space-space components are zero. *The space-time split is not a Lorentz covariant procedure.* In Sec. 4 in [17] the derivation of the correct LT is presented. If the matrix (5) in [17], $(\mathbf{E}_H)^{\mu\nu}$, is transformed in the way in which the matrix of any other bivector transforms under the LT, Eq. (13) in [17], then the matrix (12) in [17], $(\mathbf{E}'_H)^{\mu\nu}$, is obtained in which the space-space components are different from zero and *the components* $(\mathbf{E}_H)^{\mu\nu}$ *transform under the LT again to the components* $(\mathbf{E}'_H)^{\mu\nu}$; *there is no mixing with the components of the matrix of the magnetic field bivector.* In general, as shown in [15, 16] the electric and magnetic fields can be represented by different algebraic objects; vectors, bivectors or their combination.

*The correct LT always transform the 4D algebraic object representing the electric field only to the electric field; there is no mixing with the magnetic field.*

## VI. BRIEFLY ABOUT THE FIELD EQUATIONS AND THE EXPRESSIONS FOR $T(n)$, $U$ AND $S$ IN TERMS OF VECTORS $E$ AND $B$

### A. A short discussion of the field equations with vectors $E$ and $B$

If the decomposition of $F$ from (12) is introduced into (6) then the field equation (34) is obtained

$$\begin{aligned}[\partial_\alpha(\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu cB_\nu) - (j^\beta/\varepsilon_0)]\gamma_\beta + \\ \partial_\alpha(\delta^{\alpha\beta}{}_{\mu\nu}v^\mu cB^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu E_\nu)\gamma_5\gamma_\beta = 0,\end{aligned} \tag{34}$$

where $E^\alpha$ and $B^\alpha$ are the basis components in the standard basis of the 4D vectors $E$ and $B$, $\delta^{\alpha\beta}{}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$ and $\gamma_5$ is the pseudoscalar in the $\{\gamma_\mu\}$ basis. This is Eq. (40) in [16], but there it is written using some unspecified basis $\{e_\mu\}$. The first part in (34) comes from $\partial\cdot F = j/\varepsilon_0 c$ and the second one (the source-free part) comes from $\partial\wedge F = 0$. As discussed in detail in [16] Eq. (34) *is the relativistically correct, manifestly covariant field equation* that generalizes the usual Maxwell equations with the 3D fields $\mathbf{E}$ and $\mathbf{B}$. It, (34), can be compared with the usual formulation with the 3D quantities going to the $\gamma_0$-frame in which $v = c\gamma_0$ and Eq. (15) holds. This yields that Eq. (34) becomes

$$\begin{aligned}(\partial_k E^k - j^0/c\varepsilon_0)\gamma_0 + (-\partial_0 E^i + c\varepsilon^{ijk0}\partial_j B_k - j^i/c\varepsilon_0)\gamma_i + \\ (-c\partial_k B^k)\gamma_5\gamma_0 + (c\partial_0 B^i + \varepsilon^{ijk0}\partial_j E_k)\gamma_5\gamma_i = 0.\end{aligned} \tag{35}$$

The equation (35) contains all four usual Maxwell equations in the component form. The first part (with $\gamma_\alpha$) in (35) contains *two Maxwell equations* in the *component form*, the Gauss law for the electric field (the first bracket, with $\gamma_0$) and the Ampère-Maxwell law (the second bracket, with $\gamma_i$). The second part (with $\gamma_5\gamma_\alpha$) contains the *component form* of another *two Maxwell equations*, the Gauss law for the magnetic field (with $\gamma_5\gamma_0$) and Faraday's law (with $\gamma_5\gamma_i$). As explained in detail in [16] and as seen from (35) in this geometric approach the Ampère-Maxwell law and Gauss's law are inseparably connected in one law and the same happens with Faraday's law and the law that expresses the absence of magnetic charge. It is not so in the usual formulation with the 3D $\mathbf{E}$ and $\mathbf{B}$.

Observe that the usual component form of the Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned}\partial_k E_k - j^0/c\varepsilon_0 &= 0, \quad -\partial_0 E_i + c\varepsilon_{ikj}\partial_j B_k - j^i/c\varepsilon_0 = 0,\\ \partial_k B_k &= 0, \quad c\partial_0 B_i + \varepsilon_{ikj}\partial_j E_k = 0\end{aligned} \tag{36}$$

is obtained from the covariant Maxwell equations (7) using the usual identifications of six independent components of $F^{\mu\nu}$ with three components $E_i$ and three components $B_i$ as in Sec. VII, Eqs. (41) and (43). But, as shown in Sec. VII, such an identification is meaningless in the $\{r_\mu\}$ basis, which means that Maxwell equations (36) do not hold in the $\{r_\mu\}$ basis. Moreover, the components of the 3D fields from (36) transform according to the LPET (45) and not according to mathematically correct LT, MILT, (18) - (21), which causes, as explicitly shown in [16], that Eqs. (36) are not covariant under the LT. On the other hand, contrary to the formulation of the electromagnetism with the 3D **E** and **B**,

*the formulation with the 4D fields $E$ and $B$, i.e., with equation* (34), *is correct not only in the $\gamma_0$ - frame with the standard basis $\{\gamma_\mu\}$ but in all other relatively moving frames and it holds for any permissible choice of coordinates, i.e., bases.*

This consideration reveals that the 4D fields $E$ and $B$ that transform like in (18) - (21) and the field equation (34) do not have the same physical interpretation as the usual 3D fields **E** and **B** and the usual Maxwell equations (36) except in the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis in which $E^0 = B^0 = 0$.

Here, it is at place a remark about the $\gamma_0$ - frame. The dependence of the relations (13) and the field equation (34) on $v$ reflects the arbitrariness in the selection of the $\gamma_0$ - frame, but at the same time this arbitrariness makes that Eqs. (13) and (34) are independent of that choice. The $\gamma_0$ - frame can be selected at our disposal depending on the considered problem which proves that we don't have a kind of "preferred" frame theory. Some examples will be discussed in Secs. X and XI.

The generalization of the field equation for $F$ (5), *i.e.*, (6), to a magnetized and polarized moving medium with the generalized magnetization-polarization bivector $\mathcal{M}(x)$ is presented in [27]. That generalization is obtained simply replacing $F$ by $F + \mathcal{M}/\varepsilon_0$, which yields the primary equations for the electromagnetism in moving media, Eq. (7) in [27] with AQs and Eq. (8) in [27] with the CBGQs in the standard basis. It is shown in [27] that if in equation for $F$ (5) $j = j^{(C)} + j^{(\mathcal{M})}$ is the total current density then (5), *i.e.*, (6), *holds unchanged in moving medium as well*; $j^{(C)}$ is the conduction current density of the *free* charges and $j^{(\mathcal{M})} = -c\partial\cdot\mathcal{M}$ is the magnetization-polarization current density of the *bound* charges. $\mathcal{M}(x)$, in a similar way as for $F$, can be decomposed into two vectors, the polarization vector $P(x)$ and the magnetization vector $M(x)$ and the unit time-like vector $u/c$, where the vector $u$ is identified with bulk 4D velocity vector of the medium in spacetime; $\mathcal{M} = (1/2)\mathcal{M}^{\mu\nu}\gamma_\mu\wedge\gamma_\nu$, $\mathcal{M}^{\mu\nu} = (1/c)(P^\mu u^\nu - P^\nu u^\mu) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta}M_\alpha u_\beta$. Hence, the fundamental equations for moving media with the CBGQs in the $\{\gamma_\mu\}$ basis, Eqs. (29) and (30) in [27], are obtained. Their sum is the generalization of the field equation (34) to a magnetized and polarized moving medium. The equation (29) in [27] (Eq. (37) here) is the part with sources

$$\partial_\alpha\{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu + c\varepsilon^{\alpha\beta\mu\nu}v_\mu B_\nu] + [\delta^{\alpha\beta}{}_{\mu\nu}P^\mu u^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu}M_\mu u_\nu]\}\gamma_\beta = j^{(C)\beta}\gamma_\beta, \tag{37}$$

where $\delta^{\alpha\beta}{}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$. The equation (30) in [27] is the source-free part and it is the same as the source-free part in (34). As stated in [27] these equations are the fundamental equations for moving media and *they replace all usual Maxwell's equations (with 3D vectors) for moving media.*

Furthermore, in the same way as for vacuum, i.e., as in [29], one can derive from the field equation the stress-energy vector $T(n)$ for a moving medium simply replacing $F$ by $F + \mathcal{M}/\varepsilon_0$ in Eqs. (26), (37-47) in [29], i.e., in the equations (9) that are given in Sec. III. B. here. The expression for $T(n)$, $T(n) = Un + (1/c)S$, will remain unchanged, but the energy density $U$ and the Poynting vector $S$ will change according to the described replacement. This will be important in the discussion of Abraham-Minkowski controversy.

### B. The expressions for $T(n)$, $U$ and $S$ in terms of vectors $E$ and $B$

Inserting the decomposition of $F$ (10) into the coordinate-free expressions for the stress-energy vector $T(n)$, the energy density $U$, the Poynting vector $S$ (and other quantities) that are found in [29] and given in Sec. IIIB here, we can express them in terms of 1-vectors $E$ and $B$. This most general form for $T(n)$ and the quantities derived from $T(n)$ will not be presented here, but the special form in which it is taken that $v = cn$ as in [34]. In that case $T(n)$ takes very simple form

$$T(n) = (-\varepsilon_0/2)(E^2 + c^2B^2)n + \varepsilon_0 c\gamma_5(E\wedge B\wedge n). \tag{38}$$

Again, as before, the first term in (38) ($n-\parallel$ ) yields the energy density $U$ as $U = n\cdot T(n) = (-\varepsilon_0/2)(E^2 + c^2B^2)$

and the second term ($n-\perp$, i.e., $n\cdot S = 0$) is $(1/c)$ of the Poynting vector $S$. The coordinate-free momentum density $g$ is defined as before $g = (1/c^2)S$ and the angular-momentum density is $M = (1/c)T(n)\wedge x = (1/c)U(n\wedge x) + g\wedge x$, where $T(n)$ is given by the relation (38).

All these quantities can be written in some basis $\{e_\mu\}$, e.g., $\{\gamma_\mu\}$, $\{r_\mu\}$, $\{\gamma'_\mu\}$, etc. bases, as CBGQs. Thus $T(n)$ (38) becomes

$$T(n) = (-\varepsilon_0/2)(E^\alpha E_\alpha + c^2 B^\alpha B_\alpha)n^\lambda e_\lambda + \varepsilon_0 c\widetilde{\varepsilon}^\lambda{}_{\alpha\beta}E^\alpha B^\beta e_\lambda, \tag{39}$$

where $\widetilde{\varepsilon}_{\lambda\alpha\beta} = \varepsilon_{\rho\lambda\alpha\beta}n^\rho$ is the totally skew-symmetric Levi-Civita pseudotensor induced on the hypersurface orthogonal to $n$. The energy density $U$ in the $\{e_\mu\}$ basis is determined by the first term in (39) $U = (-\varepsilon_0/2)(E^\alpha E_\alpha + c^2 B^\alpha B_\alpha)$, and the Poynting vector $S$ in the $\{e_\mu\}$ basis is determined by the second term in (39) as $S = c\varepsilon_0\widetilde{\varepsilon}^\lambda{}_{\alpha\beta}E^\alpha B^\beta e_\lambda$. Of course from (39) one can easily find $g$ and $M$ in the $\{e_\mu\}$ basis.

Although we don't need the energy-momentum tensor $T^{\mu\nu}$ (which is defined in the $\{e_\mu\}$ basis as $T^{\mu\nu} = T^\mu \cdot e^\nu$) we quote here $T^{\mu\nu}$ expressed in terms of components of 4D vectors $E$ and $B$ in some basis $\{e_\mu\}$ as

$$\begin{aligned} T^{\mu\nu} = \varepsilon_0[(g^{\mu\nu}/2 - n^\mu n^\nu)(E^\alpha E_\alpha + c^2 B^\alpha B_\alpha) - (E^\mu E^\nu + c^2 B^\mu B^\nu) + \\ (\varepsilon^{\mu\alpha\beta\lambda}n_\lambda n^\nu + \varepsilon^{\nu\alpha\beta\lambda}n_\lambda n^\mu)cB_\alpha E_\beta], \end{aligned} \tag{40}$$

see also [34] and [35] in which the $\{\gamma_\mu\}$ basis is used. It has to be emphasized once again that, in contrast to all earlier definitions, our definitions of $T(n)$, $U$, $S$, $g$ and $M$ are the definitions of the Lorentz invariant quantities.

One can compare these expressions with familiar ones from the 3D space considering our definitions in the $\gamma_0$ - frame (the standard basis $\{\gamma_\mu\}$ and $v = c\gamma_0$) and consequently $E^0 = B^0 = 0$. Then $U$ takes the familiar form $U = (-\varepsilon_0/2)(E^i E_i + c^2 B^i B_i)$, $i = 1, 2, 3$. (Observe that the chosen metric is $g_{\mu\nu} = diag(1, -1, -1, -1)$ and that the components $E^i$ are identified with the components of the 3D $\mathbf{E}$, e.g., $E^1 = E_x$, which means that $E^i E_i$ corresponds to $-(E_x^2 + E_y^2 + E_z^2)$ in terms of the components of the 3D $\mathbf{E}$.) Similarly, in the $\gamma_0$ - frame, the Poynting vector becomes the familiar expression $S = \varepsilon_0 c^2 \varepsilon_0{}^i{}_{jk}E^j B^k \gamma_i$, $i, j, k = 1, 2, 3$, whence one also easily finds $g$ and $M$ in the $\gamma_0$ - frame. Notice that all quantities in these expressions are well-defined quantities on the 4D spacetime. This again nicely illustrates our main idea that *from the ISR viewpoint in the 4D spacetime the 3D quantities don't exist by themselves but only as well-defined 4D quantities taken in a particular - but otherwise arbitrary - inertial frame of reference, here the $\gamma_0$ - frame.*

## VII. THE DERIVATION OF THE LPET OF THE 3D E AND B IN JACKSON [2]

Einstein's derivation [1] of the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$ is discussed in [6]. Here, we shall first discuss the derivation of the LPET from the usual covariant approaches, e.g., from [2]. There the covariant form of the Maxwell equations (7) is written with $F^{\alpha\beta}$ and its dual ${}^*F^{\alpha\beta}$, where ${}^*F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. As already said, in order to get the component form of the Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$ (36) from equation (7) one simply makes *the identification of* the six independent components of $F^{\alpha\beta}$ with six components of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$. These identifications are

$$E_i = F^{i0}, \qquad B_i = (1/2)\varepsilon_{ijk}F_{kj} \tag{41}$$

(the indices $i$, $j$, $k$, ... $= 1, 2, 3$), Eq. (11.137) in [2], e.g., $E_x = E_1 = F^{10}$. The components of the 3D fields $\mathbf{E}$ and $\mathbf{B}$ are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric $\varepsilon$ tensor too. The super- and subscripts are used only on the components of the 4D quantities. The 3D $\mathbf{E}$ and $\mathbf{B}$ are *geometric quantities in the 3D space* and they are constructed from these six independent components of $F^{\mu\nu}$ and *the unit 3D vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, e.g., $\mathbf{E} = F^{10}\mathbf{i} + F^{20}\mathbf{j} + F^{30}\mathbf{k}$. Observe that $F^{\alpha\beta}$ is not a tensor since $F^{\alpha\beta}$ are only components implicitly taken in the standard basis. The components are coordinate quantities and they do not contain the whole information about the physical quantity, since a basis of the spacetime is not included. In the covariant approaches, e.g., [2], one transforms by the passive LT the covariant Maxwell equations (7) and finds

$$\partial'_\alpha F'^{\alpha\beta} = j'^\beta/\varepsilon_0 c, \quad \partial'_\alpha {}^*F'^{\alpha\beta} = 0. \tag{42}$$

Then, it is supposed that the same identification of the components as in equation (41) holds for a relatively moving inertial frame $S'$, i.e., for the transformed components $E'_i$ and $B'_i$

$$E'_i = F'^{i0}, \quad B'_i = (1/2c)\varepsilon_{ijk}F'_{kj}. \tag{43}$$

The same remark about the (generic) subscripts holds also here. The components $F^{\alpha\beta}$ transform under the LT as, e.g.,

$$F'^{10} = F^{10}, \; F'^{20} = \gamma(F^{20} - \beta F^{21}), \; F'^{30} = \gamma(F^{30} - \beta F^{31}), \tag{44}$$

which yields (by equations (41) and (43)) that

$$E'_1 = E_1,\ E'_2 = \gamma(E_2 - \beta cB_3),\ E'_3 = \gamma(E_3 + \beta cB_2), \tag{45}$$

what is equation (11.148) in [2]. Thus, in such approaches, e.g., [2], *the LPET of the components of* $\mathbf{E}$ *and* $\mathbf{B}$ *are derived assuming that they transform under the LT as the components of* $F^{\alpha\beta}$ *transform.*

However, from the mathematical viewpoint, i.e., from the ISR viewpoint, there are several objections to the mathematical correctness of such a procedure. Some of them are the following:

1) As seen, e.g., from Sec. 3.1 in [10], such an identification of the components of $\mathbf{E}$ and $\mathbf{B}$ with the components of $F^{\alpha\beta}$ is synchronization dependent and, particularly, it is meaningless in the "radio," "r" synchronization, i.e., in the $\{r_\mu\}$ basis, see [6] and below.

2) The 3D vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{E}'$, $\mathbf{B}'$ are constructed in both frames in the same way, i.e., multiplying the components, e.g., $E_{x,y,z}$ and $E'_{x,y,z}$ by the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, respectively. This procedure gives the LPET of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, Eq. (11.149) in [2]. But, as seen from (48), *the components* $F^{\alpha\beta}$ *are multiplied by the bivector basis* $\gamma_\alpha \wedge \gamma_\beta$ *and not by the unit 3D vectors. In the 4D spacetime the unit 3D vectors are ill-defined algebraic quantities and there are no LT, or some other transformations, that transform the unit 3D vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *into the unit 3D vectors* $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$.

3) As mentioned above (the objection 1)) the identification of the components of $\mathbf{E}$ and $\mathbf{B}$ with the components of $F^{\alpha\beta}$, (41), is synchronization dependent. If the components $F^{\alpha\beta}$ of $F$ are transformed by the transformation matrix $R^\mu_{\ \nu}$ to the $\{r_\mu\}$ basis, then it is obtained that, e.g.,

$$F^{10}_r = F^{10} - F^{12} - F^{13}. \tag{46}$$

Hence, as shown, e.g., in [6] and [10], in the $\{r_\mu\}$ basis the identification $E_{1r} = F^{10}_r$, as in (41), yields that the component $E_{1r}$ is expressed as the combination of $E_i$ and $B_i$ components from the $\{\gamma_\mu\}$ basis

$$E_{1r} = F^{10}_r, \quad E_{1r} = E_1 + cB_3 - cB_2. \tag{47}$$

This means that *if the "r" synchronization is used then it is not possible to make the usual identifications* (41) *and* (43).

4) As discussed in [29] and in Sec. IIIB here in the 4D geometric approach, i.e., in the ISR, the primary quantity for the whole electromagnetism is a physically measurable quantity, the bivector field $F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, where $\gamma_\mu \wedge \gamma_\nu$ is the bivector basis and the basis components $F^{\mu\nu}$ are determined as $F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F$. In the same way as for any other CBGQ it holds that bivector $F$ *is the same 4D quantity* for relatively moving inertial observers and for all bases chosen by them, e.g.,

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)F^{\mu\nu}_r r_\mu \wedge r_\nu = (1/2)F'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu = (1/2)F'^{\mu\nu}_r r'_\mu \wedge r'_\nu, \tag{48}$$

where, as in (26), the primed quantities in both bases $\{\gamma_\mu\}$ and $\{r_\mu\}$ are the Lorentz transforms of the unprimed ones. *Only the whole* $F$ *from* (48) *is a mathematically correctly defined quantity* and it does have a definite physical reality. The components $F^{i0}$, or $F^{ij}$ (implicitly determined in the standard basis $\{\gamma_\mu\}$), if taken alone, are not properly defined physical quantities in the 4D spacetime. The transformations of these components, e.g., Eq. (44), which are extracted from the LT of the whole properly defined physical quantity $F = (1/2)F^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta$, are not the mathematically correct LT. They do not refer to *the same 4D quantity* for relatively moving observers. Hence, from the ISR viewpoint, the determination of $\mathbf{E}$ and $\mathbf{B}$ by the components $F^{i0}$ and $F^{ij}$, respectively, as the quantities that do not depend on the 4D velocity of the observer is not mathematically correct. In contrast to it, the determination of vectors $E$ and $B$ relative to the observer by the decomposition of $F$, i.e., by Eqs. (10) and (11) with coordinate-free quantities, or (12) and (13) with the CBGQs is mathematically correct. Every antisymmetric tensor of the second rank (as a geometric quantity) can be decomposed into two vectors and a unit timelike vector, in this case, $v/c$. The components are coordinate quantities and they are only a part of the representation in some basis of an abstract, coordinate-free bivector $F$.

5) In addition, it is worth mentioning that in the usual covariant approaches, e.g., [2], the components $F^{\alpha\beta}$ are defined in terms of a 4D vector potential $A^\alpha = (\Phi, \mathbf{A})$, Eq. (11.132) in [2], as $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, Eq. (11.136) in [2]. The 3D fields $\mathbf{E}$ and $\mathbf{B}$ are determined in terms of the potentials by Eq. (11.134) in [2], which, together with Eq. (11.136) in [2], leads to Eq. (11.137) in [2] in which, as already stated, the components $F^{\alpha\beta}$ are expressed in terms of the components of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$. According to that procedure from [2] the 4D vector potential $A^\alpha$ (gauge dependent and thus unmeasurable quantity) is considered to be the primary quantity which determines the measurable quantities, the electric and magnetic fields and also $F^{\alpha\beta}$. Observe that, contrary to the assertions from

[2], $A^\alpha$ is not a 4D vector. $A^\alpha$ are only components implicitly taken in the standard basis of the 4D vector $A = A^\mu\gamma_\mu$. In the 4D spacetime only the whole 4D potential $A = A^\mu\gamma_\mu = A^\mu_r r_\mu$ is a well-defined quantity, whereas it is not the case with the usual scalar potential $\Phi$ and the 3D vector potential $\mathbf{A}$ in which the components $A_{x,y,z}$ are multiplied by the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and not by the properly defined unit 4D vectors $\gamma_\mu$.

Similar derivations of the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$ are given in many other textbooks, e.g., in "R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics* Volume II (Addison-Wesley, Reading, 1964)" in Sec. 26-3 under the title "Relativistic transformation of the fields," in "L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon, New York, 1975 )" in Sec. 24 under the title "Lorentz transformation of the field", etc.. All objections 1) - 5) from this section hold in the same measure for the mentioned well-known textbooks.

In Sec. 12.3.2 in [22] under the title "How the Fields Transform" the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$ (only components implicitly taken in the standard basis) Eq. (12. 109) in [22], are derived using the Lorentz contraction, the time dilation and the 3D fields. But, as discussed in Sec. I and in Appendix here, the Lorentz contraction and the time dilation are ill-defined in the 4D spacetime; they are synchronization dependent and consequently they are not intrinsic relativistic effects. That derivation from [22] is the same as in "E.M. Purcell, *Electricity and Magnetism*, 2nd ed. (McGraw-Hill, New York, 1985)". The derivation of the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$ from Purcell's textbook is discussed at great length and objected from the ISR viewpoint in Sec. 4.3. in [6] and will not be repeated here.

## VIII. THE DERIVATION OF THE LPET OF THE 3D E AND B IN BLANDFORD AND THORNE [21]

As mentioned in the Introduction *the nature of electric and magnetic fields is discussed* in Sec. 1.10 in [21]. There, it is concluded that *these fields are the 4D fields.* If one applies the LT to BT's equation (1.109) (it is our Eq. (13)), e.g., to the electric field 4D vector then, as discussed above, *both* $F^{\alpha\beta}$ *and* $w_\beta$ (their $w$ is our $v$) have to be transformed. The equation (19) would be obtained and Eq. (26) would hold. This is *not* noticed by Blandford and Thorne, [21], and they believe as all others that their Eq. (1.113) with the 3D vectors (the same as Eq. (11.149) in [2], i.e., Eq. (1) here) is the mathematically correct "*Relationship Between Fields Measured by Different Observers.*" Thus, although they deal with 4D GQs they still consider that in the 4D spacetime, in the same way as in the 3D space, the 3D vectors are the physical quantities, whereas the 4D quantities are considered to be only mathematical, auxiliary, quantities. This is visible in the treatment of the Lorentz force in [21]. In the usual formulations the physical meaning of 3D vectors $\mathbf{E}$ and $\mathbf{B}$ is determined by the Lorentz force as a 3D vector $\mathbf{F}_L$=$q\mathbf{E} + q\mathbf{u}\times\mathbf{B}$ and by Newton's second law $\mathbf{F} = d\mathbf{p}/dt$, $\mathbf{p}$ =$m\gamma_u\mathbf{u}$. BT start with the correct equation (1.106) ($dp^\mu/d\tau = (q/c)F^{\mu\nu}u_\nu$, our notation), but then instead of to use the decomposition of $F^{\mu\nu}$, their Eq. (1.110), our Eq. (12), but only components $F^{\mu\nu}$, *they deal with the usual identification of the components (in the standard basis) of* $F^{\mu\nu}$ *with the components of the 3D vectors* $\mathbf{E}$ *and* $\mathbf{B}$, their Eq. (1.107), our Eq. (41), which, as discussed above, is synchronization dependent and even meaningless in the $\{r_\mu\}$ basis, see Eqs. (46) and (47). Finally they get "the familiar Lorentz-force form" in terms of *the 3D vectors* $\mathbf{E}$ and $\mathbf{B}$, their Eq. (1.108). Thus, the same as in the usual approaches. It is interesting that Thorne and Blandford (TB) applied the same consideration about the Lorentz force as above in their recent very good textbook [36] that is written in geometric terms.

However, in the 4D spacetime, as mentioned above, the Lorentz force $K_L$ is given by Eq. (8) in terms of $F$ and $u$. Using the decomposition of $F$ (10) the Lorentz force $K_L$ becomes

$$K_L = (q/c)\left[(1/c)E\wedge v + (IB)\cdot v\right]\cdot u, \tag{49}$$

where $u$ is the velocity vector of a charge $q$ (it is defined to be the tangent to its world line). Note that *there are two velocity vectors in* $K_L$ *if it is expressed in terms of fields* $E$ *and* $B$, because $E$ and $B$ are determined relative to the observer with velocity vector $v$. If $K_L$ is represented as a CBGQ in the standard basis it is

$$K_L = K^\mu_L\gamma_\mu = (q/c)F^{\mu\nu}u_\nu\gamma_\mu = (q/c)\{[(1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\lambda\mu\nu\rho}v_\lambda B_\rho]u_\nu\}\gamma_\mu, \tag{50}$$

where $F^{\mu\nu}$ is from Eq. (12). In contrast to the usual expression for the Lorentz force with the 3D fields $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{F}_L$=$q\mathbf{E} + q\mathbf{u}\times\mathbf{B}$, the Lorentz force with the 4D fields $E$ and $B$ (49) or (50) contains not only the 4D velocity $u$ of a charge $q$ but also the 4D velocity $v$ of the observer who measures 4D fields. It can be simply checked that for $K^\mu_L\gamma_\mu$ (50) the relation (51) holds

$$K_L = K^\mu_L\gamma_\mu = K'^\mu_L\gamma'_\mu = K^\mu_{Lr}r_\nu = K'^\mu_{Lr}r'_\nu \tag{51}$$

as for any other 4D CBGQ. *In the 4D spacetime, the physical meaning of* $E^\mu$ *and* $B^\mu$ *is determined by the Lorentz force* $K_L$ (49), i.e., $K^\mu_L\gamma_\mu$ (50) *and by the 4D expression for Newton's second law*

$$K^\mu_L\gamma_\mu = (dp^\mu/d\tau)\gamma_\mu, \quad p^\mu = mu^\mu, \tag{52}$$

$p^\mu$ is the proper momentum (components) and $\tau$ is the proper time. *All components $E^\mu$ and $B^\mu$, thus $E^0$ and $B^0$ as well, are equally well physical and measurable quantities by means of the mentioned $K_L^\mu$ (50) and the 4D expression for Newton's second law* (52). Hence, in the 4D spacetime, contrary to the assertion from [21, 36], the use of the mathematically correct 4D GQs as in (49) or (50) cannot lead to "the familiar Lorentz-force form."

Furthermore, BT in [21] (and TB in [36]), state: "Only after making such an observer-dependent "3+1 split" of spacetime into space plus time do the electric field and magnetic field come into existence as separate entities." But, as shown above, in the 4D spacetime "3+1 split" is ill-defined. It does not hold in the $\{r_\mu\}$ basis and even in the $\{\gamma_\mu\}$ basis *it is not a Lorentz covariant procedure*, i.e., the 3-surface of simultaneity for one observer (with 4D velocity $w$) cannot be transformed by the LT into the 3-surface of simultaneity for a relatively moving inertial observer (with 4D velocity $w'$). If for one observer $w^\mu = (1,0,0,0)$ then for a relatively moving inertial observer it holds that $w'^\mu = (\gamma, -\beta\gamma, 0, 0)$). Hence, it cannot be mathematically correct that *both* $E^0_w = 0$ and $E^0_{w'} = 0$, but it is necessary $E^0_{w'} \neq 0$, as in (19) or (33). This means that their, [21], Eq. (1.107), our Eq. (41), is not correct. It does not follow from Eq. (1.109), our Eq. (13) (without unit 4D vectors). Also, Eq. (1.113) cannot be obtained by a mathematically correct procedure from Eq. (1.110). Simply, in the 4D spacetime there is no room for the 3D quantities; an independent physical reality has to be consistently attributed to the 4D GQs and not to the usual 3D quantities. Obviously, an important statement from Chapter 1 in [21] that is already mentioned above: "*We shall state physical laws, e.g. the Lorentz force law, as geometric, coordinate-free relationships between these geometric, coordinate free quantities,*" has to be changed in this way:

*In the 4D spacetime physical laws, e.g. the Lorentz force law, are geometric, coordinate-free relationships between the 4D geometric, coordinate free quantities.*

The 3D fields $\mathbf{E}$ and $\mathbf{B}$ and the Lorentz force $\mathbf{F}_L$ ($\mathbf{F}_L = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$) are also geometric quantities but in the 3D space, which means that they do not have well-defined mathematical and physical meaning in the 4D spacetime.

In addition, BT in [21] (TB in [36]) consider, as almost the whole physics community, that the Lorentz contraction and the time dilation are the intrinsic relativistic effects. However, as already mentioned, in [6], [7] and in Appendix here, it is exactly proved that such an opinion is not correct since both the Lorentz contraction and the time dilation are ill-defined in the 4D spacetime. Instead of them the 4D GQs, the position 4D vector, the distance 4D vector between two events and the spacetime length have to be used, since they are properly defined quantities in the 4D spacetime

## IX. ADDITIONAL REMARKS ABOUT THE 4D LORENTZ FORCE

Here, it is at place to give some additional comments about the Lorentz force $K_L$ (49) or (50) as a 4D GQ. It is visible from (49) or (50) that the Lorentz force ascribed by an observer comoving with a charge, $u = v$, i.e., if the charge and the observer world lines coincide, then $K_L$ is purely electric, $K_L = qE$. In the general case when $u$ is different from $v$, i.e. when the charge and the observer have distinct world lines, $K_L$ (49) or (50) can be written in terms of $E$ and $B$ as a sum of the $v$ - orthogonal part, $K_{L\perp}$ ($K_{L\perp} \cdot v = 0$) and $v$ - parallel part, $K_{L\parallel}$ ($K_{L\parallel} \wedge v = 0$). As the CBGQs they are

$$\begin{aligned} K_L &= K_{L\perp} + K_{L\parallel}, \quad K_{L\perp} = (q/c^2)[(v^\nu u_\nu)E^\mu + \varepsilon^{\lambda\mu\nu\rho} v_\lambda u_\nu c B_\rho]\gamma_\mu, \\ K_{L\parallel} &= (q/c^2)[-(E^\nu u_\nu)v^\mu]\gamma_\mu. \end{aligned} \tag{53}$$

Speaking in terms of the prerelativistic notions one can say that in the approach with the vectors $E$ and $B$ the $v$ - orthogonal part, $K_{L\perp}$, from (53) plays the role of the usual Lorentz force lying on the 3D hypersurface orthogonal to $v$, whereas $K_{L\parallel}$ from (53) is related to the work done by the field on the charge. This can be seen specifying (53) to the $\gamma_0$ - frame, $v = c\gamma_0$, in which $E^0 = B^0 = 0$. In the $\gamma_0$ - frame it is possible to compare the 4D vector $K_L$ with the usual 3D Lorentz force, $\mathbf{F}_L$=$q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$, which yields

$$\begin{aligned} K^0_L\gamma_0 &= K^0_{L\parallel}\gamma_0 = -(q/c)E^i u_i \gamma_0, \quad K^0_{L\perp} = 0, \\ K^i_L\gamma_i &= K^i_{L\perp}\gamma_i = q((E^i + \varepsilon^{0ijk} u_j B_k)\gamma_i, \quad K^i_{L\parallel}\gamma_i = 0 \end{aligned} \tag{54}$$

It is visible from (54) that $K^0_L$ is completely determined by $K_{L\parallel}$, whereas the spatial components $K^i_L$ are determined by $K_{L\perp}$. However, as already mentioned several times, in this 4D geometric approach, i.e., in the ISR, *only both parts taken together, i.e., the whole $K_L = K_{L\perp} + K_{L\parallel}$ does have a definite physical meaning and it defines the 4D Lorentz force both in the theory and in experiments.*

In Sec. 2.5 in [29], under the title "The Lorentz force and the motion of charged particle in the electromagnetic field $F$" the definition of $K_L$ in terms of $F$ is exclusively used ($K_L = (q/c)F \cdot u$) without introducing the electric and magnetic fields. Observe that the 4D GQs $K$ ($K_L$), $p$, $u$ transform in the same way, like any other 4D vector,

i.e., according to the LT and not according to the awkward usual transformations of the 3D force **F**, e.g., Eqs. (12.65) - (12.67) in [22], and the 3D momentum **p**, i.e., the 3D velocity **u**. In the first paper in [37], under the title "Four Dimensional Geometric Quantities versus the Usual Three-Dimensional Quantities: The Resolution of Jackson's Paradox," it is shown that only with the use of the 4D Lorentz force (49), (50) or (53), the torque bivector $N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, $N^{\mu\nu} = x^\mu K_L^\nu - x^\nu K_L^\mu$ and the angular momentum bivector $M = (1/2)M^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, $M^{\mu\nu} = m(x^\mu u^\nu - x^\nu u^\mu)$ there is no apparent electrodynamic paradox with the torque and that the principle of relativity is naturally satisfied. The second paper in [37] is a simpler version of the first one. The mentioned paradox is described in [38] and it consists in the fact that there is a 3D torque **N** and thus $d\mathbf{L}/dt$ ($\mathbf{N} = d\mathbf{L}/dt$) in one inertial frame, but no 3D angular momentum $\mathbf{L}'$ and no 3D torque $\mathbf{N}'$ in another relatively moving inertial frame. Similar electrodynamic paradoxes with the 3D torque appear in the Trouton-Noble paradox, see, e.g., [39] and the "charge-magnet paradox" [40]. Using the above mentioned 4D GQs, 4D Lorentz force, the torque and angular momentum bivectors it is explicitly shown in [41], [29] for the Trouton-Noble paradox and [42], [10] for Mansuripur's paradox that there is no paradox and consequently there is no need for some "resolutions" of the paradoxes, e.g., by the introduction of the Einstein-Laub force, [40], or by the introduction of some "hidden" quantities, e.g., [43].

## X. THE ELECTROMAGNETIC FIELD OF A POINT CHARGE IN UNIFORM MOTION

It is worth mentioning that the majority of physicists consider that if the electric field would be transformed by the LT again into the electric field as in (19) then it would imply that moving electrons produce no magnetic field. In Sec. 5.6 in [42] the electromagnetic field of a point charge in uniform motion is treated in detail. There it is shown that the formulation of that problem with the 4D fields and their MILT (18), (19) is mathematically completely correct but its physical interpretation is different than in the usual formulation with the 3D fields and their LPET. The consideration presented in 5.6.2 - 5.6.2.2 in [42] explicitly shows that the formulation with the 4D fields that transform according to the MILT (18), (19) simply explains the existence of the electric *and magnetic fields* for a moving electron.

### A. The bivector field $F$

Here we shall briefly quote the main results from [42]. *In the 4D formulation the primary quantity is the the bivector field $F$*. The expression for $F$ for an arbitrary motion of a point charge is given in [29] by Eqs. (10) (coordinate-free quantities) and (11) (CBGQs). Particularly, for a charge $Q$ moving with constant 4D velocity vector $u$, $F$ is given by Eq. (12) in [29] (coordinate-free quantities), i.e., Eq. (65) in [42]

$$F(x) = G(x \wedge (u/c)), \quad G = kQ/\left|x \wedge (u/c)\right|^3, \tag{55}$$

where $k = 1/4\pi\varepsilon_0$. $G$ is a number, a Lorentz scalar. The geometric character of $F$ is contained in $x \wedge (u/c)$. If that $F$ is written as a CBGQ in the standard basis it is

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma; F^{\mu\nu} = G(1/c)(x^\mu u^\nu - x^\nu u^\mu), G = kQ/[(x^\mu u_\mu)^2 - c^2 x^\mu x_\mu]^{3/2}. \tag{56}$$

In order to find the explicit expression for $F$ from (56) in the $S'$ frame in which the charge $Q$ is at rest one has simply to put into (56) that $u = c\gamma'_0$ with $\gamma_0'^{\mu} = (1, 0, 0, 0)$. Then, $F = (1/2)F'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu$ and

$$F = F'^{i0}(\gamma'_i \wedge \gamma'_0) = Gx'^i(\gamma'_i \wedge \gamma'_0), \quad G = kQ/(x'^i x'_i)^{3/2}. \tag{57}$$

In $S'$ and in the standard basis, the basis components $F'^{\mu\nu}$ of the bivector $F$ are obtained from (56) and they are:

$$F'^{i0} = -F'^{0i} = kQx'^i/(x'^i x'_i)^{3/2}, \quad F'^{ij} = 0. \tag{58}$$

In the charge's rest frame there are only components $F'^{i0}$, which are the same as the usual components of the 3D electric field **E** for a charge at rest.

In the same way we find the expression for $F$ (56) in the $S$ frame in which the charge $Q$ is moving, i.e., $u = u^\mu\gamma_\mu$ with $u^\mu/c = (\gamma, \gamma\beta, 0, 0)$. Then

$$\begin{aligned} F = G\gamma[(x^1 - \beta x^0)(\gamma_1 \wedge \gamma_0) + x^2(\gamma_2 \wedge \gamma_0) + x^3(\gamma_3 \wedge \gamma_0) \\ -\beta x^2(\gamma_1 \wedge \gamma_2) - \beta x^3(\gamma_1 \wedge \gamma_3)], \quad G = kQ/[\gamma^2(x^1 - \beta x^0)^2 + (x^2)^2 + (x^3)^2]^{3/2}. \end{aligned} \tag{59}$$

In $S$ and in the standard basis, the basis components $F^{\mu\nu}$ of the bivector $F$ are again obtained from (56) and they are

$$\begin{aligned} F^{10} &= G\gamma[(x^1-\beta x^0),\ F^{20}=G\gamma x^2,\ F^{30}=G\gamma x^3, \\ F^{21} &= G\gamma\beta x^2,\ F^{31}=G\gamma\beta x^3,\ F^{32}=0. \end{aligned} \tag{60}$$

The expression for $F$ as a CBGQ in the $S$ frame can be find in another way as well, i.e., to make the LT of the quantities from (57). Observe that the CBGQs from (57) and (59), which are the representations of the bivector $F$ in $S'$ and $S$ respectively, are equal, $F$ from (57) $=$ $F$ from (59); *they are the same quantity $F$ from* (55), i.e., (56), *for observers in $S'$ and $S$*. It can be seen from (60) that $F^{i0}$ *and* $F^{ij}$ are different from zero for a moving charge and they are the same as the usual components of the 3D fields **E** and **B**, respectively. But, as already discussed and as seen from (48) and (56) only the whole $F$, which contains components *and the bivector basis*, is properly defined physical quantity.

## B. The expressions for the 4D $E$ and $B$

### 1. The general expressions

From the known $F$ (56) and the relations (13) we can construct in a mathematically correct way the 4D vectors $E$ and $B$ for a charge $Q$ moving with constant velocity $u$. If written as CBGQs in the standard basis they are given by Eq. (73) in [42]

$$\begin{aligned} E &= E^\mu\gamma_\mu = (G/c^2)[(u^\nu v_\nu)x^\mu - (x^\nu v_\nu)u^\mu]\gamma_\mu, \\ B &= B^\mu\gamma_\mu = (G/c^3)\varepsilon^{\mu\nu\alpha\beta}x_\nu u_\alpha v_\beta\gamma_\mu, \end{aligned} \tag{61}$$

where $G$ is from (56). *The vectors $E$ and $B$ are explicitly observer dependent*, i.e., dependent on $v$. For the same $F$ the vectors $E$ and $B$ will have different expressions depending on the velocity of observers who measure them. It is visible from (61) that $E$ and $B$ depend on *two* velocity 4D vectors $u$ and $v$, whereas the usual 3D vectors **E** and **B** depend only on the 3-velocity of the charge $Q$. Note also that although $E$ and $B$ as the CBGQs from (61) depend not only on $u$ but on $v$ as well the electromagnetic field $F$ from (56) does not contain the velocity of the observer $v$. This result directly proves that *the electromagnetic field $F$ is the primary quantity from which the observer dependent $E$ and $B$ are derived.* The expressions for $E$ and $B$ from (61) correctly describe fields in all cases simply specifying $u$ *and* $v$ and this assertion holds not only for the $\{\gamma_\mu\}$ basis but for the $\{r_\mu\}$ basis as well, i.e., the relation like (26) holds for the expressions from (61). However, observe that, as already mentioned several times, *the 4D fields $E$ and $B$ and the usual 3D fields* **E** *and* **B** *have the same physical interpretation only in the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis in which $E^0 = B^0 = 0$.* In Sec. 5.6.2.1 in [42] the general expression (61) for the 4D $E$ and $B$ is specified to the case when the $\gamma_0$ - frame is the rest frame of the charge $Q$, the $S'$ frame, $v = c\gamma'_0 = u$, whereas in Sec. 5.6.2.2 the same is made in the case when the $\gamma_0$ - frame is the laboratory frame, the $S$ frame, $v = c\gamma_0$, in which the charge $Q$ is moving, $u^\mu = (\gamma c, \gamma\beta c, 0, 0)$.

### 2. The $\gamma_0$ - frame is the rest frame of the charge $Q$, the $S'$ frame

If the $\gamma_0$ - frame is the $S'$ frame, $v = c\gamma'_0 = u$, then (61) yields that $B = 0$ and only an electric field (Coulomb field) remains, which is in agreement with the usual 3D formulation. Hence, it follows from (61) that

$$E = E'^i\gamma'_i = Gx'^i\gamma'_i, \quad E'^0 = 0, \quad G = kQ/(x'^i x'_i)^{3/2}; \quad B = B'^\mu\gamma'_\mu = 0. \tag{62}$$

The components in (62) agree, as it is expected, with the usual result with the 3D fields, e.g., with components in Eq. (11) in the first paper in [40]. Now comes the essential difference relative to all usual approaches. In order to find the representations of $E$ and $B$ in $S$, i.e., the CBGQs $E^\mu\gamma_\mu$ and $B^\mu\gamma_\mu$, we can either perform the LT of $E'^\mu\gamma'_\mu$ and $B'^\mu\gamma'_\mu$ that are given by (62), or simply to take in (61) that *both* the charge $Q$ *and the "fiducial" observers* are moving relative to the observers in $S$; $v^\mu = u^\mu = (\gamma c, \gamma\beta c, 0, 0)$. This yields Eq. (63) ((75) in [42]), i.e., the CBGQs $E^\mu\gamma_\mu$ and $B^\mu\gamma_\mu$ in $S$ with the condition that the "fiducial" observers are in $S'$, $v = c\gamma'_0$, which is the rest frame of the charge $Q$, $u = c\gamma'_0$,

$$\begin{aligned} E = E^\mu\gamma_\mu = G[&\beta\gamma^2(x^1-\beta x^0)\gamma_0 + \gamma^2(x^1-\beta x^0)\gamma_1 + \\ &x^2\gamma_2 + x^3\gamma_3], \quad B = B^\mu\gamma_\mu = 0, \end{aligned} \tag{63}$$

where $G$ is that one from (59). The result (63) significantly differs from the result obtained by the LPET, Eqs. (12a), (12b) in [40]. Under the LT, i.e., the MILT, the electric field vector transforms again to the electric field vector and the same for the magnetic field vector. It is worth mentioning that, in contrast to the conventional results, it holds that $E'^{\mu}\gamma'_{\mu}$ from (62) is $= E^{\mu}\gamma_{\mu}$ from (75) in [42]; *they are the same quantity E for all relatively moving inertial observers.* The same holds for $B$, $B'^{\mu}\gamma'_{\mu}$ from (62) is $= B^{\mu}\gamma_{\mu}$ from (63) and they are $= 0$ for all observers. Furthermore, observe that in $S'$ there are only the spatial components $E'^{i}$, whereas in $S$, as seen from (63), there is also the temporal component $E^0$ as a consequence of the LT.

### 3. The $\gamma_0$ - frame is the laboratory frame, the $S$ frame

Now, let us take that the "fiducial" observers are in $S$, $v = c\gamma_0$, in which the charge $Q$ is moving, $u^{\mu} = (\gamma c, \gamma\beta c, 0, 0)$. In contrast to the previous case, *both E and B* are different from zero. The expressions for the CBGQs $E^{\mu}\gamma_{\mu}$ and $B^{\mu}\gamma_{\mu}$ in $S$ can be simply obtained from (61) taking in it that $v = c\gamma_0$ and $u^{\mu} = \gamma c\gamma_0 + \gamma\beta c\gamma_1$. This yields that $E^0 = B^0 = 0$ (from $v = c\gamma_0$) and the spatial parts are

$$\begin{aligned} E &= E^i\gamma_i = G\gamma[(x^1 - \beta x^0)\gamma_1 + x^2\gamma_2 + x^3\gamma_3], \\ B &= B^i\gamma_i = (G/c)[0\gamma_1 - \gamma\beta x^3\gamma_2 + \gamma\beta x^2\gamma_3], \end{aligned} \tag{64}$$

where $G$ is again as in (59). The 4D vector fields $E$ and $B$ from (64) can be compared with the usual expressions for the 3D fields $\mathbf{E}$ and $\mathbf{B}$ of an uniformly moving charge, e.g., from Eqs. (12a), (12b) in [40]. It is visible that they are similar, but $E$ and $B$ in (64) are the 4D fields and all quantities in (64) are correctly defined in the 4D spacetime, which transform by the LT, i.e., the MILT, whereas the fields in Eqs. (12a), (12b) in [40] are the 3D fields that transform according to the LPET.

In order to find the representations of $E$ and $B$ in $S'$, i.e., the CBGQs $E'^{\mu}\gamma'_{\mu}$ and $B'^{\mu}\gamma'_{\mu}$, we can either perform the LT of $E^{\mu}\gamma_{\mu}$ and $B^{\mu}\gamma_{\mu}$ that are given by (64), or simply to take in (61) that relative to $S'$ the "fiducial" observers are moving with $v = v'^{\mu}\gamma'_{\mu}$, $v'^{\mu} = (c\gamma, -\gamma\beta c, 0, 0)$, and the charge $Q$ is at rest relative to the observers in $S'$, $u'^{\mu} = (c, 0, 0, 0)$. This yields the CBGQs $E'^{\mu}\gamma'_{\mu}$ and $B'^{\mu}\gamma'_{\mu}$ in $S'$ with the condition that the "fiducial" observers are in $S$, $v = c\gamma_0$,

$$\begin{aligned} E &= E'^{\mu}\gamma'_{\mu} = G\gamma[-\beta x'^{1}\gamma'_0 + x'^{1}\gamma'_1 + x'^{2}\gamma'_2 + x'^{3}\gamma'_3], \\ B &= B'^{\mu}\gamma'_{\mu} = (G/c)[0\gamma'_0 + 0\gamma'_1 - \gamma\beta x'^{3}\gamma'_2 + \gamma\beta x'^{2}\gamma'_3], \end{aligned} \tag{65}$$

where $G$ is as in (62). Again, as in the case that $v = c\gamma'_0$, it holds that $E^{\mu}\gamma_{\mu}$ from (64) is $= E'^{\mu}\gamma'_{\mu}$ from (65); they are the same quantity $E$ for all relatively moving inertial observers. The same holds for $B^{\mu}\gamma_{\mu}$ from (64) which is $= B'^{\mu}\gamma'_{\mu}$ from (65) and *they are both different from zero.* Note that in this case there are only the spatial components $E^i$ in $S$, whereas in $S'$ there is also the temporal component $E'^0$ as a consequence of the MILT.

It is visible from (65) that if the $\gamma_0$ - frame is the lab frame ($v = c\gamma_0$) in which the charge $Q$ is moving then $E'^{\mu}\gamma'_{\mu}$ and $B'^{\mu}\gamma'_{\mu}$ in the rest frame of the charge $Q$, the $S'$ frame, are completely different than those from (62); in (65) $B'^{\mu}\gamma'_{\mu}$ is different from zero and the representation of $E$ contains also the term $E'^0\gamma'_0$.

It has to be emphasized that *all four expressions for E and B*, (62), (63), (64) *and* (65), *are the special cases of E and B given by* (61). *They all give the same F from* (56), *which is the representation* (*CBGQ*) *of F given by the basis free, abstract, bivector* (55).

## XI. COMPARISON WITH EXPERIMENTS

It is usually considered that the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$, Eq. (11.149) in [2], i.e., (1) here, are firmly confirmed by experiments and, accordingly, that there are not separate electric and magnetic 4D vectors. In particular, it is considered that in the rest frame of the charge the 3D electric field is given, e.g., by Eq. (11) in the first paper in [40], the 3D magnetic field is zero, whereas in a relatively moving inertial frame the 3D vectors are given by Eqs. (12a), (12b) in [40] and they are obtained, as stated in [40], by the LT of the electric field of the point charge from $S'$ to $S$ frame. In [40], these LT, are what we call the LPET of the 3D vectors, Eq. (11.149) in [2], i.e., (1) here. However, note that the 3D fields (11) in [40] for a charge at rest and (12a), (12b) in [40] for an uniformly moving charge are usually obtained as the solutions of Maxwell's equations without the use of the LT. In that case both 3D fields are determined in the same frame, usually it is the laboratory frame, but they refer to a charge at rest in that frame and to an uniformly moving charge in the same frame. All experiments are made *only* in the laboratory frame in which the fields are measured for the two mentioned states of motion. The observers are in both cases *only* in the laboratory frame. Hence it is not true that the LPET of the 3D $\mathbf{E}$ and $\mathbf{B}$ fields are firmly confirmed by experiments. As seen

from the preceding sections if the observers are at rest in the laboratory frame and they use the standard basis, i.e., the lab frame is the $\gamma_0$ - frame, then $E^0 = B^0 = 0$ and the spatial components of the 4D $E$ and $B$ agree with the components of the 3D $\mathbf{E}$ and $\mathbf{B}$ for both, a charge at rest in the lab frame, equation (62) and an uniformly moving charge in the lab frame, equation (64). This shows that in the cases in which only the fields are investigated the 4D fields are in the same agreement with *existing* experiments as are the 3D fields. From that result one could think that the 3D fields can explain all experiments and that there is no need for the 4D fields.

However, as shown previously, the formulation with the 4D GQs is in a true agreement, independent of the chosen inertial reference frame and of the chosen system of coordinates in it, with experiments in electromagnetism, the motional emf in [15, 27, 29], the Faraday disk in [16] and the Trouton-Noble experiment in [29, 41]. *As shown in the mentioned papers* [15, 16, 27, 29, 41] *it is not the case with the usual 3D formulation.*

Thus, for example, in section 5.1 in [15] the motional emf $\varepsilon$ is calculated using the 3D quantities (the Lorentz force as a 3D vector, $\mathbf{F}_L{=}q\mathbf{E}+q\mathbf{u}\times\mathbf{B}$, and $\varepsilon = \oint(\mathbf{F}_L/q)\cdot\mathbf{dl}$, Eq. (26) in [15]) and their LPET, Eq. (11.149) in [2], i.e., (1) here. In section 5.2 in [15] $\varepsilon$ is calculated using the 4D GQs and their mathematically correct LT, i.e., the MILT, like (19). The Lorentz force $K_L$ is defined by equations (49) or (50). *The emf $\varepsilon$ is defined as an invariant 4D quantity, the Lorentz scalar*, Eq. (35) in [14], $\varepsilon = \int_\Gamma (K_L/q)\cdot dl = \int_\Gamma (K_L^\mu/q) dl_\mu = (1/c)\int_\Gamma F^{\mu\nu}u_\nu dl_\mu$, where *vector dl* is the infinitesimal spacetime length and $\Gamma$ is the spacetime curve. In section 5.1 in [15] it is shown that the emf obtained by the application of the LPET is different for relatively moving 4D observers, $\varepsilon = VBl$ in $S$ (the laboratory frame) and $\varepsilon' = \gamma VBl$ in $S'$, Eqs. (27) and (29) respectively, which means that *the principle of relativity is not satisfied in the usual formulation of electromagnetism with the 3D quantities and their LPET* of $\mathbf{E}$ and $\mathbf{B}$. On the other hand, if the 4D GQs and their MILT, like (19), are used then the emf is always the same; it is independent of the chosen reference frame and of the chosen system of coordinates in it. Thus, if $\varepsilon$ is defined as an invariant 4D quantity, the Lorentz scalar, Eq. (35) in [15], then always the same value for $\varepsilon$ is obtained, $\varepsilon = \gamma VBl$, Eqs. (36) and (37) in [15]. These results unambiguously show that *the principle of relativity is naturally satisfied in the approach with 4D GQs and their mathematically correct MILT*, like (19). The result that the conventional theory with the 3D $\mathbf{E}$ and $\mathbf{B}$ and their LPET, Eqs. (11.148) and (11.149) in [2], i.e., (1) here, yields different values for the motional emf $\varepsilon$ for relatively moving inertial observers, whereas the approach with 4D GQs and their MILT yields always the same value for $\varepsilon$, $\varepsilon = \gamma VBl$, is very strong evidence that the approach with 4D GQs is a relativistically correct approach.

*It is for the experimentalists to find the way to measure the emf $\varepsilon$ with a great precision in order to see that in the laboratory frame $\varepsilon = \gamma VBl$ and not simply $\varepsilon = VBl$.*

Such an experiment would be a crucial experiment that could verify from the experimental viewpoint the validity of the formulation of the electromagnetism with the 4D GQs and their mathematically correct LT, MILT, (18) - (21). The same result as in Sec. 5.2 in [15] is obtained in [27] but exclusively dealing with $F$ and not with its decompositions (10) and (12). As mentioned above the same difference between the usual approach with 3D fields and the approach with 4D GQs is shown to exist in the case of the Faraday disk in [16].

Furthermore, as already mentioned in Sec. IX in the approach with 4D GQs and their MILT, like (19) (the Trouton-Noble paradox [29, 41], Jackson's paradox [37] and the "charge-magnet paradox" [42]) there is no paradox and thus there is no need for some resolutions of the paradoxes and there is no need for the introduction of some "hidden" quantities [43] or the Einstein-Laub force [40].

As already stated, in [28], the constitutive relations and the magnetoelectric effect in moving media are explained in a completely new way using 4D GQs and their mathematically correct MILT. In equation (17) in [28] it is shown how the polarization vector $P(x)$ depends on $E$, $B$, $u$, the bulk velocity vector of the medium and $v$, the velocity vector of the observer who measures fields, $P^\mu\gamma_\mu = (\varepsilon_0\chi_E/c)[(1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta}v_\alpha B_\beta]u_\nu\gamma_\mu$, whereas in equation (18) in [28] the same is shown for the magnetization vector $M(x)$, $M^\mu\gamma_\mu = \varepsilon_0\chi_B[(B^\mu v^\nu - B^\nu v^\mu) + (1/c)\varepsilon^{\mu\nu\alpha\beta}E_\alpha v_\beta]u_\nu\gamma_\mu$. The last term in the expression for $P^\mu\gamma_\mu$ and the last term in the expression for $M^\mu\gamma_\mu$ describe the magnetoelectric effect in a moving dielectric. According to the last term in $P^\mu\gamma_\mu$ a moving dielectric becomes electrically polarized if it is placed in a magnetic field, the Wilsons' experiment, reference [29] in [28]. Similarly, the last term in $M^\mu\gamma_\mu$ shows that a moving dielectric becomes magnetized if it is placed in an electric field, Röntgen's experiment, reference [30] in [28].

## XII. DISCUSSION AND CONCLUSION

The main point in the whole paper is that in the 4D spacetime physical laws are geometric, coordinate-free relationships between the 4D geometric, coordinate-free quantities. This point of view is also adopted in the nice textbook [21] (and, as well, in [36]) but not in the consistent way. They still introduce the 3D vectors and their transformations, e.g., in Sec. 1.10 in [21] and this is discussed in Sec. VIII here. A fully consistent application of this viewpoint is adopted in Oziewicz's papers, see, e.g., [25]. The same viewpoint is adopted in all my papers given in the references, including the present paper. Particularly, in [44], under the title "Nature of Electric and Magnetic Fields; How the

Fields Transform" we have already presented many results that are given in this paper. Here, in this paper, the mathematically correct proofs are given that in this geometric approach, i.e., in the ISR, the electric and magnetic fields are properly defined vectors on the 4D spacetime, Secs. IIIA and IIIC. According to Oziewicz's proof from Sec. IIIA, e.g., the electric field vector must have four components (some of them can be zero) since it is defined on the 4D spacetime and not, as usually considered, only three components. In Sec. IIIC it is taken into account that, as proved in [29], the primary quantity for the whole electromagnetism is the electromagnetic field bivector $F$. The decomposition of $F$ given by Eq. (10) expresses $F$ in terms of observer dependent electric and magnetic 4D vectors $E$ and $B$, which are given by Eq. (11). Both, Eqs. (10) and (11), are with the abstract, coordinate-free quantities. This is in a sharp contrast with the usual covariant approaches, e.g., [2, 22] in which it is considered that $F^{\alpha\beta}$ (the components implicitly taken in the standard basis) is physically well-defined quantity. Moreover, these components are considered to be six indepent components of the 3D **E** and **B**, see Eqs. (41) and (43). Then, as described in Sec. VII, in these approaches [2, 22], *the transformations of the components of* **E** *and* **B** (45) *are obtained supposing that they transform under the LT as the components of* $F^{\alpha\beta}$ *transform*, Eqs. (44) and (45). The objections to such treatment are given in Sec. VII, the objections 1) - 5). From the mathematical viewpoint all these objections are well-founded since they are based on the following facts:

1) The bivector $F(x)$, as described in detail in [29] and very briefly in Sec. III.B here, is determined, for the given sources, by the solutions of the equation (5), i.e., (6) (with CBGQs in the $\{\gamma_\mu\}$ basis) and not by the components of the 3D **E** and **B**. It is a 4D GQ and not only components. It yields a complete description of the electromagnetic field without the need for the introduction either the field vectors or the potentials.

2) As seen from Sec. II and particularly from Eqs. (46) and (47) the identification of the components of the 3D **E** and **B** with the components of $F^{\alpha\beta}$ is synchronization dependent. Moreover, it is completely meaningless in the "r" synchronization, i.e., in the $\{r_\mu\}$ basis. Both bases, the commonly used standard basis with Einstein's synchronization and the $\{r_\mu\}$ basis with the "r" synchronization are equally well physical and relativistically correct bases. It is worth mentioning that in [45], in which a geometric approach with exterior forms is used, it is considered that the usual identification, (41), is premetric, but, as explained above, it is synchronization dependent and thus dependent on the chosen metric of the 4D spacetime. This is discussed in detail in Sec. 5.3 in [28] in connection with the constitutive relations.

Furthermore, it is proved in Sec. IV.A with the coordinate-free quantities and the active LT and in Sec. IV.B with CBGQs and the passive LT that the mathematically correct LT, the MILT, of, e.g., the electric field vector are given by (18) - (21) and not by the LPET of the 3D vectors Eqs. (11.148) and (11.149) in [2], i.e., Eq. (45) or Eq. (22) here.

In Sec. V.A the same fundamental difference between the correct LT, the MILT, and the LPET of the 3D vectors is explicitly exposed using matrices. The equations (29) - (33) refer to the correct LT, the MILT, of the components in the standard basis of the electric field 4D vector in which the transformed components $E'^{\mu}$ are obtained as $E'^{\mu} = c^{-1}F'^{\mu\nu}v'_{\nu}$, i.e., *both* $F^{\mu\nu}$ *and the velocity of the observer* $v = c\gamma_0$ *are transformed* by the matrix of the LT $A^{\mu}_{\nu}$ (the boost in the direction $x^1$). It is visible from Eq. (33) that the same components are obtained as $E'^{\mu} = A^{\mu}_{\nu}E^{\nu}$ and they are the same as in (19). This means that *under the mathematically correct LT, the MILT, the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms.* As stated at the end of Sec. V.A if $E$ is written as a CBGQ then again holds the relation (26) as for any other CBGQ. On the other hand Eq. (27) refers to the LPET in which the transformed components $E'^{\mu}_{F}$ are obtained as $E'^{\mu}_{F} = c^{-1}F'^{\mu\nu}v_{\nu}$, i.e., only $F^{\mu\nu}$ *is transformed by the LT but not the velocity of the observer* $v = c\gamma_0$. These transformed components $E'^{\mu}_{F}$ are the same as in Eq. (22). The transformed spatial components $E'^{i}_{F}$ are the same as are the transformed components of the usual 3D vector **E**, i.e., as in Eq. (11.148) in [2]. However, according to these transformations the 4D vector with $E^0 = 0$ is transformed in such a way that the transformed temporal component is again zero, $E'^{0}_{F} = 0$. Hence, as stated in Sec. V.A, such transformations cannot be the mathematically correct LT.

It can be concluded from the whole consideration in this paper that in the 4D spacetime an independent physical reality has to be attributed to the 4D geometric quantities, coordinate-free quantities or the CBGQs, e.g., the electromagnetic field bivector $F$, the 4D vectors of the electric $E$ and magnetic $B$ fields, etc., and not to the usual 3D quantities, e. g., the 3D **E** and **B**. This is the answer to the question what is the nature of the electric and magnetic fields. Furthermore, the mathematically correct LT are properly defined on the 4D spacetime. They can correctly transform only the 4D quantities like $E$ and $B$, the mathematically correct LT, the MILT, (18) - (21), according to which, e.g., *the electric field 4D vector transforms again only to the electric field 4D vector as any other 4D vector transforms.* The LT cannot act on the 3D quantities like the 3D **E** and **B**, which means that the LPET of the 3D quantities, e.g., the 3D vectors **E** and **B**, Eqs. (11.148) and (11.149) in [2], i.e., Eq. (45) or Eq. (22) here, are not properly defined LT in the 4D spacetime. This is the answer to the question how the fields transform.

## APPENDIX

In this Appendix we briefly describe the essential differences between the 4D geometric approach, the ISR, and Einstein's definition of the Lorentz contraction, e.g., for a moving rod. This is explained in detail in Secs. 2 - 2.3 in [8] and Secs. 3.1, 4.1 and Figs. 1 and 3 in [6]. Here, the mathematical formalism is different than in [8] and [6]. In the geometric approach one deals with the abstract 4D geometric quantities, i.e., with the position vectors $x_A$, $x_B$, of the events $A$ and $B$, respectively, with the distance vector $l_{AB} = x_B - x_A$ and with the spacetime length, $l = L_0$, see (67). The essential feature of the geometric approach is that *any abstract 4D geometric quantity*, e.g., the distance vector $l_{AB} = x_B - x_A$, *is only one quantity, the same quantity in the 4D spacetime* for all relatively moving frames of reference and for all systems of coordinates that are chosen in them. The abstract vector $l_{AB}$ can be decomposed in different bases and then these representations, the CBGQs, of the same abstract 4D geometric quantity $l_{AB}$ contain both the basis components and the basis vectors. Let us explain it taking a particular choice for $l_{AB}$, which in the usual "3+1" picture corresponds to a rod that is at rest in an inertial frame of reference (IFR) $S$ (with the standard basis in it) and situated along the common $x^1$, $x'^1$ − axes. Its rest length is denoted as $L_0$. The situation is depicted in Fig. 1 in [6]. $l_{AB}$ is decomposed, i.e., it is written as a CBGQ, in the standard basis and in $S$ and $S'$, where the rod is moving, as

$$l_{AB} = l^{\mu}_{AB}\gamma_{\mu} = 0\gamma_0 + L_0\gamma_1 = l'^{\mu}_{AB}\gamma'_{\mu} = -\beta\gamma L_0\gamma'_0 + \gamma L_0\gamma'_1, \tag{66}$$

As already stated several times, the components $l^{\mu}_{AB}$ are transformed by the LT and the basis vectors $\gamma_{\mu}$ by the inverse LT leaving the whole CBGQ unchanged. In $S$, the position vectors $x_{A,B}$ are determined simultaneously, $x^0_B - x^0_A = l^0_{AB} = 0$, i.e., the temporal part of $l^{\mu}_{AB}$ is zero. In the standard basis, which is commonly used in the usual approaches, there is a dilation of the spatial part $l'^1_{AB} = \gamma L_0$ with respect to $l^1_{AB} = L_0$ and not the Lorentz contraction as predicted in Einstein's formulation of SR. Similarly, as explicitly shown in [8] and [6], in the $\{r_{\mu}\}$ basis, i.e., with the "r" synchronization, if only spatial parts of $l^{\mu}_{AB,r}$ and $l'^{\mu}_{AB,r}$ are compared then one finds the dilation $\infty \succ l'^1_{AB,r} \geq L_0$ for all $\beta_r$. However, the comparison of only spatial parts of the components of the distance vector $l_{AB}$ in $S$ and $S'$ is physically meaningless in the geometric approach, since *some components of the tensor quantity, when they are taken alone, do not correspond to some definite 4D physical quantity.* Note that if $l^0_{AB} = 0$ then the LT yield that $l'^{\mu}_{AB}$ in any other IFR $S'$ contains the time component as well, $l'^0_{AB} = x'^0_B - x'^0_A = -\beta\gamma L_0 \neq 0$. Hence, *the LT yield that the spatial ends of the rod are not determined simultaneously in $S'$, i.e., the temporal part of $l'^{\mu}_{AB}$ is not zero.* For the spacetime length $l$ it holds that

$$l^2 = \mid l^{\mu}_{AB} l_{AB,\mu} \mid = \mid l'^{\mu}_{AB} l'_{AB,\mu} \mid = \mid l^{\mu}_{AB,r} l_{AB,r,\mu} \mid = L^2_0. \tag{67}$$

In $S$, the rest frame of the rod, where the temporal part of $l^{\mu}_{AB}$ is $l^0_{AB} = 0$, the spacetime length $l$ is a measure of the spatial distance, i.e., of the rest spatial length of the rod, as in the prerelativistic physics. *The observers in all other IFRs will "look" at the same events $A$ and $B$, the same distance vector $l_{AB}$ and the same spacetime length $l$, but associating with them different coordinates; it is the essence of the geometric approach. They all obtain the same value $l$ for the spacetime length, $l = L_0$.*

It is worth mentioning, once again, that the 4D geometric treatment with $l_{AB}$ and $l$ is a generalization and a mathematically better founded formulation of the ideas expressed by Rohrlich [11] and Gamba [46]. Indeed, Rohrlich [11] states: "A quantity is therefore physically meaningful (in the sense that it is of the same nature to all observers) if it has tensorial properties under Lorentz transformations." Similarly Gamba [46], when discussing the sameness of a physical quantity (for example, *a nonlocal quantity* $A_{\mu}(x_{\lambda}, X_{\lambda})$, which is a function of two points in the 4D spacetime $x_{\lambda}$ and $X_{\lambda}$) for different inertial frames of reference $S$ and $S'$, declares: "The quantity $A_{\mu}(x_{\lambda}, X_{\lambda})$ for $S$ is the same as the quantity $A'_{\mu}(x'_{\lambda}, X'_{\lambda})$ for $S'$ when all the primed quantities are obtained from the corresponding unprimed quantities through Lorentz transformations (tensor calculus)." Rohrlich and Gamba worked with the usual covariant approach, i.e., with the components implicitly taken in the standard basis, which means that only Einstein's synchronization is considered to be physically admissible. The quantities $A_{\mu}(x_{\lambda}, X_{\lambda})$ and $A'_{\mu}(x'_{\lambda}, X'_{\lambda})$ refer to the same physical quantity, but they are not mathematically equal quantities since bases are not included. In the approach with the 4D geometric quantities, i.e., in the ISR, one deals with mathematically equal quantities, e.g., for a nonlocal quantity $l_{AB} = x_B - x_A$ it holds that

$$l_{AB} = l^{\mu}_{AB}\gamma_{\mu} = l'^{\mu}_{AB}\gamma'_{\mu} = l^{\mu}_{AB,r} r_{\mu} = l'^{\mu}_{AB,r} r'_{\mu} = .., \tag{68}$$

where the primed quantities are the Lorentz transforms of the unprimed ones. In order to treat different systems of coordinates on an equal footing we have derived a form of the LT that is independent of the chosen system of coordinates, including different synchronizations, see Eq. (2) in [8], or Eq. (1) in [6]. Also, Eq. (4) in [6], it is presented the transformation matrix that connects Einstein's system of coordinates with another system of coordinates in the same reference frame.

On the other hand, as shown in Sec. 2.2 in [8] and Sec. 4.1 and Fig. 3 in [6], in Einstein's formulation of SR, instead of to work with geometric quantities $x_{A,B}$, $l_{AB}$ and $l$ one deals only with the spatial, or temporal, *components* of their coordinate representations $x^\mu_A$, $x^\mu_B$ and $l^\mu_{AB}$ *in the standard basis*. The geometric character of physical quantities, i.e., the basis vectors, and some asymmetric synchronization, e.g., the "r" synchronization, which is equally physical as the Einstein synchronization, are never taken into account. According to Einstein's definition [1] of the spatial length the spatial ends of the rod must be taken simultaneously for the observer, i.e., *he defines length as the spatial distance between two spatial points on the (moving) object measured by simultaneity in the rest frame of the observer.* In the 4D (here, for simplicity, as in [8] and [6], we deal only with 2D) spacetime and in the $\{\gamma_\mu\}$ basis the simultaneous events $A$ and $B$ (whose spatial parts correspond to the spatial ends of the rod) are the intersections of $x^1$ axis (that is along the spatial basis vector $\gamma_1$) and the world lines of the spatial ends of the rod that is at rest in $S$ and situated along the $x^1$ axis. The components of the distance vector are $l^\mu_{AB} = x^\mu_B - x^\mu_A = (0, L_0)$; for simplicity, it is taken that $t_B = t_A = a = 0$. Then in $S$, the rest frame of the object, the spatial part $l^1_{AB} = L_0$ of $l^\mu_{AB}$ is considered to define the rest spatial length. Furthermore, one uses the inverse LT to express $x^\mu_A$, $x^\mu_B$ and $l^\mu_{AB}$ in $S$ in terms of the corresponding quantities in $S'$, in which the rod is moving. This procedure yields

$$\begin{aligned} l^0_{AB} &= ct_B - ct_A = \gamma(l'^0_{AB} + \beta l'^1_{AB}), \\ l^1_{AB} &= x^1_B - x^1_A = \gamma(l'^1_{AB} + \beta l'^0_{AB}). \end{aligned} \tag{69}$$

Now, instead of to work with 4D tensor quantities and their LT, as in the 4D geometric approach, in the usual formulation one forgets about the transformation of the temporal part $l^0_{AB}$, the first equation in (69), and considers only the transformation of the spatial part $l^1_{AB}$, the second equation in (69). Furthermore, *in that relation for* $l^1_{AB}$ *one assumes that* $t'_B = t'_A = t' = b$, i.e., *that* $x'^1_B$ *and* $x'^1_A$ *are simultaneously determined at some arbitrary* $t' = b$ in $S'$. However, *in 4D* (at us 2D) spacetime such an assumption means that *in* $S'$ *one does not consider the same events* $A$ *and* $B$ *as in* $S$ but some other two events $C$ and $D$, which means that $t'_B = t'_A$ has to be replaced with $t'_D = t'_C = b$. The events $C$ and $D$ are the intersections of the line (the hypersurface $t' = b$ with arbitrary $b$) parallel to the spatial axis $x'^1$ (which is along the spatial base vector $\gamma'_1$) and of the above mentioned world lines of the spatial end points of the rod. Then, in the above transformation for $l^1_{AB}$ (69) one has to write $x'^1_D - x'^1_C = l'^1_{CD}$ instead of $x'^1_B - x'^1_A = l'^1_{AB}$. The spatial parts $l^1_{AB}$ and $l'^1_{CD}$ are the *spatial distances* between the events $A$, $B$ and $C$, $D$, respectively. *In Einstein's formulation, the spatial distance* $l^1_{AB} = x^1_B - x^1_A = L_0$ *defines the spatial length of the rod at rest in* $S$, *whereas* $l'^1_{CD} = x'^1_D - x'^1_C$ *is considered to define the spatial length of the moving rod in* $S'$. Hence, from the equation for $l^1_{AB}$ (69) one finds the relation between $l'^1 = l'^1_{CD}$ and $l^1 = l^1_{AB} = L_0$ as the famous formula for the Lorentz contraction of the moving rod

$$l'^1 = x'^1_D - x'^1_C = L_0/\gamma = (x^1_B - x^1_A)/\gamma, \text{ with } t'_C = t'_D, \text{ and } t_B = t_A, \tag{70}$$

where $\gamma = (1-\beta^2)^{-1/2}$, $\beta = U/c$ and $U = |\mathbf{U}|$; $\mathbf{U}$ is the 3-velocity of $S'$ relative to $S$. As can be nicely seen from Fig. 3 in [6], the spatial lengths $L_0$ and $l'^1_{CD}$ refer not to the same 4D tensor quantity, as in the 4D geometric approach, see Fig. 1 in [6], but to two different quantities, *two different set of events in the 4D spacetime*. These quantities are obtained by the same measurements in $S$ and $S'$; the spatial ends of the rod are measured simultaneously at some $t = a$ in $S$ and also at some $t' = b$ in $S'$. But $a$ in $S$ and $b$ in $S'$ are not related by the LT or any other coordinate transformation. This means that the Lorentz contraction, as already shown by Rohrlich [11] and Gamba [46], is a typical example of an "apparent" transformation. *It has nothing in common with the LT of the 4D geometric quantities.* We see that in Einstein's approach [1] the spatial and temporal parts of events are treated separately, and moreover the time component is not transformed in the transformation that is called - the Lorentz contraction. In addition, as can be seen from Sec. 4.1. and Fig. 3 in [6], in Einstein's approach [1] the considered effect is dependent on the chosen synchronization. If the "r" synchronization is used, then there is not only the usual Lorentz contraction of the moving rod but also a length dilation depending on $\beta_r$. Thus, contrary to the generally accepted opinion, *the Lorentz contraction is not a well-defined relativistic effect in the 4D spacetime*. As seen from Fig. 4 in [6] the similar conclusion holds for the usual time dilation of the moving clock. The relativistically, i.e., mathematically, correct treatments of a moving rod and a moving clock are presented in Figs. 1 and 2 in [6].

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