GHOST PROBLEMS FROM PAULI-VILLARS TO FOURTH-ORDER QUANTUM GRAVITY AND THEIR RESOLUTION

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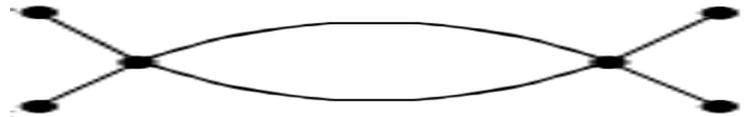
1 PAULI-VILLARS REGULATOR AND THE SEVENTY YEAR GHOST PROBLEM



In order to regularize the divergences in loop graphs and also to implement gauge invariance Pauli and Villars (RMP 1949) proposed that one replace the standard one-particle $1/(k^2 - M^2)$ propagator by the two-particle

$$D(k) = \frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2}, \tag{1}$$

so that a $1/k^2$ asymptotic ultraviolet behavior at large k^2 would be replaced by the more convergent $1/k^4$, with the quadratically divergent one loop scalar field graph only being log divergent. While Pauli and Villars recognized this as a mathematical procedure, they did not want to rule out that it might be physical. As conceived, the two particles would be associated with an action of the form

$$I_{S_1} + I_{S_2} = \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \tfrac{1}{2}M_1^2\phi_1^2 - \lambda\phi_1^4\right] + \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \tfrac{1}{2}M_2^2\phi_2^2 - \lambda\phi_2^4\right]. \tag{2}$$

with the insertion of

$$\sum |n_1\rangle\langle n_1| - \sum |n_2\rangle\langle n_2| = I, \tag{3}$$

with its ghostlike relative minus sign into

$$\langle\Omega_1|T[\phi_1(x)\phi_1(0)]|\Omega_1\rangle + \langle\Omega_2|T[\phi_2(x)\phi_2(0)]|\Omega_2\rangle \tag{4}$$

leading to (1). Thus ghosts could reduce asymptotic divergences, but at the price of **loss of probability and loss of unitarity**.

2 THE PAIS-UHLENBECK OSCILLATOR

To explore whether the Pauli-Villars regulator might be physical Pais and Uhlenbeck (PR 1950) replaced the two-field action by a one-field fourth-order derivative action

$$I_S = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - (M_1^2 + M_2^2)\partial_\mu\phi\partial^\mu\phi + M_1^2M_2^2\phi^2\right], \quad (5)$$

with fourth-order derivative equation of motion given by

$$(\partial_t^2 - \vec{\nabla}^2 + M_1^2)(\partial_t^2 - \vec{\nabla}^2 + M_2^2)\phi(x) = 0, \quad (6)$$

and associated propagator of the Pauli-Villars form

$$D(k) = \frac{1}{(k^2 - M_1^2)(k^2 - M_2^2)} = \frac{1}{(M_1^2 - M_2^2)}\left(\frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2}\right). \quad (7)$$

Since only time derivatives are relevant to quantization, on setting $\omega_1 = (\bar{k}^2 + M_1^2)^{1/2}$, $\omega_2 = (\bar{k}^2 + M_2^2)^{1/2}$ and dropping the spatial dependence, the I_S action

$$I_S = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - (M_1^2 + M_2^2)\partial_\mu\phi\partial^\mu\phi + M_1^2 M_2^2\phi^2\right], \tag{8}$$

reduces to the Pais-Uhlenbeck (PU) action

$$I_{\rm PU} = \frac{1}{2}\int dt \left[\ddot{z}^2 - \left(\omega_1^2 + \omega_2^2\right)\dot{z}^2 + \omega_1^2\omega_2^2 z^2\right]. \tag{9}$$

This is a constrained action since with only z, $\dot{z}$ and $\ddot{z}$, there are too many canonical variables for one oscillator but not enough for two. So set $x = \dot{z}$, and using the method of Ostrogradski (1850), Dirac Constraints (Mannheim and Davidson 2000, PRA 2005) or a variation of a covariantized form of the action with respect to the metric, one obtains the two-oscillator PU Hamiltonian

$$H_{\rm PU} = \frac{1}{2}p_x^2 + p_z x + \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right)x^2 - \frac{1}{2}\omega_1^2\omega_2^2 z^2, \quad [z, p_z] = i, \quad [x, p_x] = i. \tag{10}$$

Now there are no ghosts, but one now has to pay a different price: the $-\frac{1}{2}\omega_1^2\omega_2^2 z^2$ term in $H_{\rm PU}$ leads to **an energy spectrum that is unbounded from below**, the Ostrogradski (1850) instability that is characteristic of higher-derivative theories.

3 TRADING THE ENERGY INSTABILITY FOR GHOSTS

Work by Lee (PR 1954), Kallen and Pauli (MFMDVS 1955) and Heisenberg (NP 1957) reopened the ghost issue, and it was found that one could avoid negative energies in the PU theory if one quantized the PU theory with negative norm states. Specifically, if we make the standard substitutions

$$
\begin{aligned}
z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger \quad p_z = i\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\omega_1^2\omega_2(a_2 - a_2^\dagger), \\
x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \quad p_x = -\omega_1^2(a_1 + a_1^\dagger) - \omega_2^2(a_2 + a_2^\dagger),
\end{aligned} \tag{11}
$$

we obtain a Hamiltonian and commutator algebra (Mannheim and Davidson 2000, PRA 2005)

$$
\begin{aligned}
H_{\rm PU} &= 2(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + \tfrac{1}{2}(\omega_1 + \omega_2), \\
[a_1, a_1^\dagger] &= \frac{1}{2\omega_1(\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\omega_2(\omega_1^2 - \omega_2^2)},
\end{aligned} \tag{12}
$$

and note that with $\omega_1 > \omega_2$ the $[a_2, a_2^\dagger]$ commutator is **NEGATIVE**.

Also, as will become important below, we note that the $\omega_1 = \omega_2$ limit is **SINGULAR**.

TWO REALIZATIONS

$$H_{\rm PU} = 2(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + \tfrac{1}{2}(\omega_1 + \omega_2),$$
$$[a_1, a_1^\dagger] = \frac{1}{2\omega_1(\omega_1^2 - \omega_2^2)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\omega_2(\omega_1^2 - \omega_2^2)}, \quad \omega_1 > \omega_2. \tag{13}$$

If we define the vacuum according to

$$a_1|\Omega\rangle = 0, \quad a_2|\Omega\rangle = 0,$$

the energy spectrum is bounded from below with $|\Omega\rangle$ being the ground state with energy $(\omega_1 + \omega_2)/2$. But the excited state $a_2^\dagger|\Omega\rangle$, which lies at energy ω_2 **above** the ground state, has a **Dirac norm** $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$ that is **NEGATIVE**.

On the other hand if we define the vacuum according to

$$a_1|\Omega\rangle = 0, \quad a_2^\dagger|\Omega\rangle = 0,$$

the PU theory is now free of negative-norm states, but the energy spectrum is unbounded from below. (Negative energy states propagating forward in time.)

However, we note that for either realization all of the eigenvalues of $H_{\rm PU}$ are **REAL**. This will also be of importance below.

Thus the PU theory suffers from one of two twin diseases, either negative norms or negative energies. Since defining the vacuum by setting $a_2|\Omega\rangle = 0$ or by setting $a_2^\dagger|\Omega\rangle = 0$ would correspond to working in two totally different Hilbert spaces, in no single Hilbert space does one have both diseases, though in either one there is still a seemingly irrefutable problem.

4 THE LEE-WICK MODEL

In order to try to make the log divergences in QED become finite, Lee and Wick (PRD 1970) proposed to use the Pauli-Villars propagator but with a complex conjugate pair of masses

$$D_{\rm LW}(k) = \frac{1}{k^2 - M^2 + i\Gamma} - \frac{1}{k^2 - M^2 - i\Gamma} = \frac{-2i\Gamma}{(k^2 - M^2)^2 + \Gamma^2}. \tag{14}$$

With a standard Breit-Wigner being of the form

$$D_{\rm BW}(k) = \frac{1}{k^2 - M^2 + i\Gamma} = \frac{k^2 - M^2 - i\Gamma}{(k^2 - M^2)^2 + \Gamma^2} \tag{15}$$

the Lee-Wick propagator would have the same sign for its imaginary part as a Breit-Wigner, and thus be unitary. In this way Lee and Wick solve the ghost problem of the Pauli-Villars propagator, albeit with unstable rather than stable states.

5 QUANTUM GRAVITY

With Einstein gravity being based on the second-order Einstein-Hilbert action:

$$I_{\rm EH} = \frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}R^{\alpha}{}_{\alpha}, \tag{16}$$

the graviton propagator would be of the form $1/k^2$ and lead to uncontrollable quadratic divergences in perturbative quantum gravitational radiative corrections. However, a theory based on pure a fourth-order action such as the conformal gravity Weyl action (see e.g. Mannheim, 1989 - 2020), an action that is invariant under local conformal transformations of the form $g_{\mu\nu}(x)\to e^{2\alpha(x)}g_{\mu\nu}(x)$ on the metric, viz.

$$I_{\rm W} = -2\alpha_g\int d^4x\,(-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa}-\tfrac{1}{3}(R^{\alpha}{}_{\alpha})^2\right], \tag{17}$$

would lead to a $1/k^4$ propagator and thus lead to divergences that are only logarithmic and thus renormalizable. A half-way hybrid would be a combination of both second-order and fourth-order actions into $I_{\rm EH}+I_{\rm W}$. Such a combination would be of the same form as the scalar action

$$I_S \;=\; \tfrac{1}{2}\int d^4x\left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi-(M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi+M_1^2M_2^2\phi^2\right], \tag{18}$$

and thus lead to an associated propagator of the Pauli-Villars form

$$D(k)=\frac{1}{(k^2-M_1^2)(k^2-M_2^2)}=\frac{1}{(M_1^2-M_2^2)}\left(\frac{1}{k^2-M_1^2}-\frac{1}{k^2-M_2^2}\right). \tag{19}$$

Such theories are in vogue at the moment (recently there was a whole conference at CERN on the topic) as they provide a renormalizable theory of quantum gravity. (No string theory, no extra dimensions, no supersymmetry).

GREAT, BUT WHAT ABOUT THE GHOSTS? WE APPEAR TO BE AT AN IMPASSE.

6 PT SYMMETRY TO THE RESCUE – THE LEE MODEL

In 1954 Lee introduced a model in which one could do coupling constant renormalization analytically. However, the model had ghost states of negative norm and Lee, Kallen and Pauli, and Heisenberg worked very hard on the issue. However, the problem remained unsolved until the work of Bender, Brandt, Chen and Wang (PRD 2005) no less than some fifty years later.

What they found was that the ghost states only appear for a certain range of values of the renormalized coupling constant of the model and that in that range **the bare coupling constant is complex**. In consequence the theory is **not a Hermitian theory**, and one cannot use as norm or inner product **the overlap of a ket state with its Hermitian conjugate bra**. However, they found that the **theory has an antilinear PT symmetry**, and when one uses the PT theory norm, viz. **the overlap of a ket with its PT conjugate**, one finds that this norm is **POSITIVE DEFINITE**.

SOLVING THE LEE MODEL GHOST PROBLEM THIS WAY IS A CONSIDERABLE TRIUMPH FOR PT THEORY.

THUS IN GENERAL IF ONE FINDS STATES OF NEGATIVE DIRAC NORM, IT DOES NOT NECESSARILY MEAN THAT THE THEORY IS NOT UNITARY. IT COULD MEAN THAT ONE IS IN THE WRONG HILBERT SPACE AND THAT ONE IS USING THE USING THE WRONG INNER PRODUCT (THE DIRAC ONE), WITH A DIFFERENT INNER PRODUCT (THE PT ONE) BEING UNITARY.

SO WHAT ABOUT PAULI-VILLARS AND THE PAIS-UHLENBECK OSCILLATOR?

Returning now to the PU Hamiltonian

$$H_{\rm PU}=\tfrac{1}{2}p_x^2+p_zx+\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2-\tfrac{1}{2}\omega_1^2\omega_2^2z^2,\quad [z,p_z]=i,\quad [x,p_x]=i. \tag{20}$$

Now there is no complex coupling constant that could save us. However, if we set $p_z=-i\partial_z$, $p_x=-i\partial_x$, the Schrodinger equation takes the form

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}-ix\frac{\partial}{\partial z}+\frac{1}{2}(\omega_1^2+\omega_2^2)x^2-\frac{1}{2}\omega_1^2\omega_2^2z^2\right]\psi_n(z,x)=E_n\psi_n(z,x), \tag{21}$$

with the lowest positive energy state with $E_0=(\omega_1+\omega_2)/2$ having eigenfunction (Bender and Mannheim 2008)

$$\psi_0(z,x)=\exp\left[\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2z^2+i\omega_1\omega_2zx-\tfrac{1}{2}(\omega_1+\omega_2)x^2\right]. \tag{22}$$

The state $\psi_0(z,x)$ diverges as $z\to\infty$ and is thus not normalizable. The norm of the ground state

$$\langle\Omega|\Omega\rangle=\int dxdz\langle\Omega|xz\rangle\langle xz|\Omega\rangle=\int dxdz\psi_0^*(x,z)\psi_0(x,z)=\infty$$

is infinite too. Such lack of normalizability means that the closure relation

$$\sum|n_1\rangle\langle n_1|-\sum|n_2\rangle\langle n_2|=I, \tag{23}$$

could not hold as it presupposes normalizable states. However, rather than being a bad thing,

IT IS THE LACK OF NORMALIZABILITY THAT ACTUALLY SAVES THE THEORY.

To make

$$\psi_0(z,x) = \exp\left[\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\omega_1\omega_2 zx - \tfrac{1}{2}(\omega_1+\omega_2)x^2\right] \tag{24}$$

normalizable we must continue z into the complex plane. If we draw a letter X in the complex z plane, the wave function will be normalizable in a wedge (a so-called Stokes wedge) that contains the north and south quadrants in the letter X, i.e. that contains the imaginary z axis but not the real z axis (which is in the east and west quadrants).

To implement this we make a similarity transformation (Bender and Mannheim PRL 2008)

$$\begin{aligned} &y = e^{\pi p_z z/2} z e^{-\pi p_z z/2} = -iz, \quad q = e^{\pi p_z z/2} p_z e^{-\pi p_z z/2} = ip_z, \\ &[y,q] = i, \qquad [p,x] = i, \quad p_x = p, \end{aligned} \tag{25}$$

to obtain

$$e^{\pi p_z z/2} H_{\rm PU} e^{-\pi p_z z/2} = \bar{H} = \frac{p^2}{2} - iqx + \frac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2 + \frac{1}{2}\omega_1^2\omega_2^2 y^2, \tag{26}$$

$\bar{H}$ is now manifestly not Hermitian, but all of its eigenvalues are still real. It thus falls into the class of non-Hermitian theories that have a PT symmetry (x and y are PT odd and p and q are PT even) and have all energy eigenvalues real.

To make it manifest that all the eigenstates have positive norm we make an additional similarity transformation

$$Q = \alpha pq + \beta xy, \qquad \alpha = \frac{1}{\omega_1\omega_2}\log\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right), \qquad \beta = \alpha\omega_1^2\omega_2^2, \tag{27}$$

under which $\bar{H}$ transforms to (Bender and Mannheim PRL 2008)

$$e^{-Q/2}\bar{H}e^{Q/2} \;=\; \bar{H}' = \frac{p^2}{2} + \frac{q^2}{2\omega_1^2} + \frac{1}{2}\omega_1^2x^2 + \frac{1}{2}\omega_1^2\omega_2^2y^2. \tag{28}$$

We recognize $\bar{H}'$ as being a fully acceptable standard, positive norm two-dimensional oscillator system. In addition we note that with its phase being $-Q/2$ rather than $-iQ/2$, the $e^{-Q/2}$ operator is not unitary. The transformation from $\bar{H}$ to $\bar{H}'$ is thus not a unitary transformation, but is a transformation from a skew basis with eigenvectors $|n\rangle$ to an orthogonal basis with eigenvectors

$$|n'\rangle = e^{-Q/2}|n\rangle, \qquad \langle n'| = \langle n|e^{-Q/2}.$$

Then since $\langle n'|m'\rangle = \delta_{mn}$, the eigenstates of $\bar{H}$ obey

$$\langle n|e^{-Q}|m\rangle = \delta_{mn}, \quad \sum_n |n\rangle\langle n|e^{-Q} = I,$$

$$\bar{H} = \sum_n |n\rangle E_n\langle n|e^{-Q}, \quad \bar{H}|n\rangle = E_n|n\rangle, \quad \langle n|e^{-Q}\bar{H} = \langle n|e^{-Q}E_n. \tag{29}$$

We thus recognize the inner product as being not $\langle n|m\rangle$ but $\langle n|e^{-Q}|m\rangle$, with the conjugate of $|n\rangle$ being $\langle n|e^{-Q}$. This state is also the PT conjugate of $|n\rangle$, so that the inner product is the overlap of a state with its PT conjugate rather than that with its Hermitian conjugate, just as we had noted earlier. And as such this inner product is positive definite since $\langle n'|m'\rangle = \delta_{mn}$ is. The PU oscillator theory and accordingly the Pauli-Villars propagator theory are thus fully viable unitary theories.

Finally, we note that when $\omega_1 = \omega_2$ the Q operator becomes undefined.

7 BUT WHERE DID THE MINUS SIGN GO?

The propagator

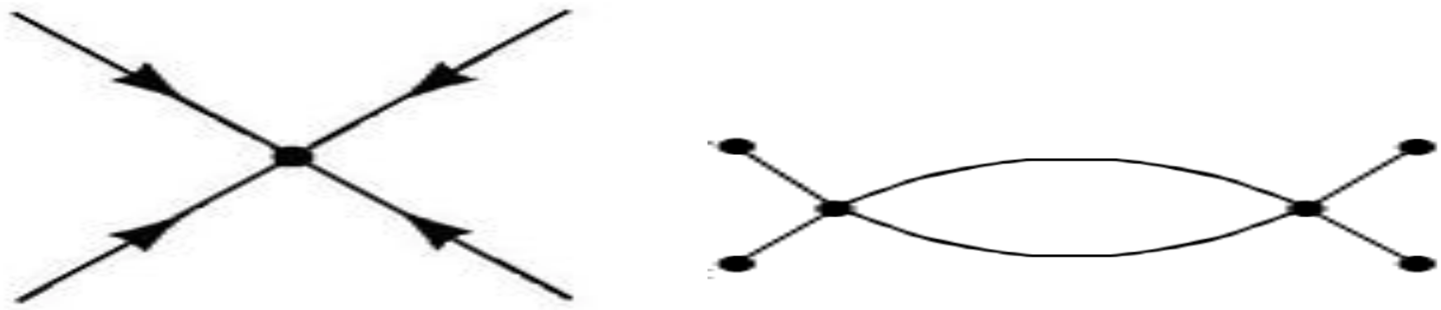
$$D(k) = \frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2} \tag{30}$$

has a relative minus sign, so where is it if all norms are positive.

The answer is that one should not identify the c-number $D(k)$ with the matrix element $\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$, but with the matrix element $\langle\Omega|e^{-Q}T[\phi(x)\phi(0)]|\Omega\rangle$ instead. Now one can insert $\sum |n\rangle\langle n|e^{-Q} = I$ into $\langle\Omega|e^{-Q}T[\phi(x)\phi(0)]|\Omega\rangle$ and generate $D(k)$ with it being the introduction of e^{-Q} that generates the minus sign (Bender and Mannheim PRD 2008) and not the presence of negative norm states.

The error was thus in associating $D(k)$ with $\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ without first having constructed the Hilbert space. You can go from the q-number Hilbert space to the c-number propagator but not the other way round. Thus we do not actually get rid of the ghost, we show that it was not there in the first place, with the reasoning that led one to think that there is a ghost being faulty.

8 BUT WHAT HAPPENS IN LOOPS?



Tree and One-Loop Graphs

With the propagator

$$D(k) = \frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2}, \tag{31}$$

some of the discontinuities across the loop graph have negative signature and thus on their own violate unitarity. However this cannot be since you cannot change the signature of a Hilbert space in perturbation theory.

What needs to be taken into consideration is that under the $e^{-Q/2}$ similarity transformation the interaction changes its form. Thus we actually cannot just treat the system as two decoupled oscillators since under the same $e^{-Q/2}$ that decouples them in H' an interaction term such as λy^4 would transform into $\lambda[y']^4$ where

$$y' \;=\; e^{-Q/2} y e^{Q/2} = y\cosh\theta + i(\alpha/\beta)^{1/2} p \sinh\theta, \quad \theta = \tfrac{1}{2}(\alpha\beta)^{1/2}. \tag{32}$$

Because of this the tree graph is no longer just λ as it would be in a Hermitian theory. Rather, the structure that it has precisely generates an additional contribution to the discontinuity that cancels the negative one in the loop, to then leave the full discontinuity positive, just as it should be (Mannheim PRD 2018).

To see how antilinearity works in general it is instructive to look at the eigenvector equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle = E|\psi(t)\rangle. \tag{33}$$

On replacing the parameter t by $-t$ and then multiplying by a general antilinear operator A we obtain

$$i\frac{\partial}{\partial t}A|\psi(-t)\rangle = AHA^{-1}A|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{34}$$

If $AHA^{-1} = H$ then energies are **real or in complex conjugate pairs** (Wigner 1931, JMP 1960).

Real energies have eigenfunctions that obey $A|\psi(-t)\rangle = \eta|\psi(t)\rangle$ with phase η. Complex conjugate pairs of energies have eigenfunctions ($|\psi(t)\rangle \sim \exp(-iEt)$ and $A|\psi(-t)\rangle \sim \exp(-iE^*t)$) that are in complex conjugate pairs with A transforming one into the other.

Mannheim JPA 2018: the necessary and sufficient condition for all eigenvalues to be real is that H has an antilinear symmetry and that its eigenstates are also eigenstates of the antilinear operator. (H being Hermitian is only sufficient for real eigenvalues.) While A could be any operator, such theories are generically called PT theories because they date back to the study of the PT symmetric $H = p^2 + ix^3$ (Bender and Boettcher PRL 1998).

10 PSEUDO-HERMITICITY

For probability conservation consider a right eigenstate $|R_i(t)\rangle$ and a candidate inner product

$$i\frac{d}{dt}|R_i(t)\rangle = H|R_i(t)\rangle, \quad i\frac{d}{dt}\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(t|(VH - H^\dagger V)|R_i(t)\rangle.$$

Probability is conserved if

$$VH = H^\dagger V, \quad VHV^{-1} = H^\dagger.$$

This is the Pseudo-Hermitian condition (Mostafazadeh JMP 2002). For the PU theory $V = e^{-Q}$ and left-eigenstate $\langle L| = \langle R|V = \langle R|e^{-Q}$.

Mannheim JPA 2018: Taking $\langle L|R\rangle$ to be time independent implies $VH = H^\dagger V$, so condition is both necessary and sufficient. Thus the most general inner product is $\langle L|R\rangle$.

But if $VHV^{-1} = H^\dagger$ then energies real or in complex pairs. Thus $VHV^{-1} = H^\dagger$ implies $AHA^{-1} = H$ and vice versa. **Pseudo-Hermiticity and antilinearity are equivalent.**

If eigenspectrum is real and complete can bring H to a Hermitian form by

$$SHS^{-1} = H' = H'^\dagger = S^{-1\dagger}H^\dagger S^\dagger, \quad S^\dagger SHS^{-1}S^{-1\dagger} = H^\dagger.$$

Thus we can identify $V = S^\dagger S$ and confirm the positivity of the left-right V norm

$$\langle L|R\rangle = \langle R|V|R\rangle = \langle R|S^\dagger S|R\rangle.$$

To make the left-right norm agree with the PT norm one should define the PT norm not as the overlap of a state with its PT conjugate but as $[|R\rangle]^{PT}\eta|R\rangle$ where η is the PT phase of the state (Mannheim PRD 2018). These phases are also the eigenvalues of the C operator introduced by Bender (RPP 2007), but these phases can exist even when there is no C operator.

With its energies just happening to be in complex conjugate pairs the Lee-Wick model is a PT theory. Since this is also the case for energies real (the PU model) in all cases the Pauli-Villars type theories are PT theories, and in all cases that is what makes them unitary and ghost free.

Currently there is much interest in the so-called Lee-Wick standard model, and it is being used for particle physics phenomenology (Grinstein, O'Connell and Wise PRD 2008). Without the practitioners knowing it, we see that PT symmetry is now being used in particle phenomenology.

12 WHY DO WE NEED TO CONTINUE INTO THE COMPLEX PLANE?

What do we mean when we say that $H = p^2 + ix^3$ is not Hermitian, and how could

$$H_{\rm PU} = \tfrac{1}{2}p_x^2 + p_z x + \tfrac{1}{2}\left(\omega_1^2 + \omega_2^2\right)x^2 - \tfrac{1}{2}\omega_1^2\omega_2^2 z^2 \tag{35}$$

not be Hermitian. We ordinarily think of position and momentum operators as being Hermitian. However that only means that when acting on their own eigenstates one can integrate by parts and throw away surface terms. However that does not mean that when acting on the eigenstates of some other operator such as the Hamiltonian that is built out of them that one can still integrate by parts. It is when there is such a mismatch that one has to continue into the complex plane into a domain where one has asymptotic convergence and one then can integrate by parts.

To see how it is you cannot tell whether a Hamiltonian might be Hermitian or not just by inspection, set $\omega_1 = \alpha + i\beta$, $\omega_2 = \alpha - i\beta$. One then obtains

$$H_{\rm PU} = \tfrac{1}{2}p_x^2 + p_z x + \left(\alpha^2 - \beta^2\right)x^2 - \tfrac{1}{2}(\alpha^2 + \beta^2)^2 z^2, \tag{36}$$

$H_{\rm PU}$ still looks Hermitian, but its eigenvalues are in a complex pair.

And even more. Set $\omega_1 = \omega_2 = \omega$. Now we have

$$H_{\rm PU} = \tfrac{1}{2}p_x^2 + p_z x + \omega^2 x^2 - \tfrac{1}{2}\omega^4 z^2. \tag{37}$$

$H_{\rm PU}$ again still looks Hermitian, but the Q operator that we introduced before

$$Q = \alpha pq + \beta xy, \qquad \alpha = \frac{1}{\omega_1\omega_2}\log\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right), \qquad \beta = \alpha\omega_1^2\omega_2^2, \tag{38}$$

which effects

$$e^{-Q/2}\bar{H}e^{Q/2} \;=\; \bar{H}' = \frac{p^2}{2} + \frac{q^2}{2\omega_1^2} + \frac{1}{2}\omega_1^2 x^2 + \frac{1}{2}\omega_1^2\omega_2^2 y^2, \tag{39}$$

becomes singular at $\omega_1 = \omega_2$ and $H_{\rm PU}$ can no longer be diagonalized. The equal frequency PU oscillator is thus a non-diagonalizable and thus non-Hermitian Jordan-block Hamiltonian that lacks a complete set of eigenstates, something that is not at all obvious from the appearance of $H_{\rm PU}$. The point $\omega_1 = \omega_2$ is the transition point (an exceptional point) between energies real and energies in a complex conjugate pair, which is why there is a singularity. This is a generic feature of antilinear symmetry.

HERMITICITY CANNOT BE ESTABLISHED BY INSPECTION, BUT ANTILINEAR SYMMETRY CAN BE.

13 IS THERE A PREFERRED ANTILINEAR SYMMETRY IN NATURE - YES CPT

The Lorentz group has a restricted sector (only real Lorentz transformations), and this sector can be extended by complex Lorentz transformations to the proper Lorentz group. On coordinates PT implements $x^\mu \to -x^\mu$, and thus so does CPT since the coordinates are charge conjugation even. With a boost in the x_1-direction implementing $x_1' = x_1 \cosh\xi + t\sinh\xi$, $t' = t\cosh\xi + x_1 \sinh\xi$, with complex $\xi = i\pi$ we obtain

$$\begin{aligned}
\Lambda^0{}_1(i\pi):&\qquad x_1 \to -x_1, \qquad t \to -t,\\
\Lambda^0{}_2(i\pi):&\qquad x_2 \to -x_2, \qquad t \to -t,\\
\Lambda^0{}_3(i\pi):&\qquad x_3 \to -x_3, \qquad t \to -t,\\
\pi\tau = \Lambda^0{}_3(i\pi)\Lambda^0{}_2(i\pi)\Lambda^0{}_1(i\pi):&\qquad x^\mu \to -x^\mu.
\end{aligned} \tag{40}$$

Bender RPP 2007: Complex $\pi\tau$ implements the linear part of a PT (and consequently CPT) transformation on coordinates.

With $\Lambda^0{}_i(i\pi)$ implementing $e^{-i\pi\gamma^0\gamma_i/2} = -i\gamma^0\gamma_i$ for Dirac gamma matrices, on introducing

$$\hat{\pi}\hat{\tau} = \hat{\Lambda}^0{}_3(i\pi)\hat{\Lambda}^0{}_2(i\pi)\hat{\Lambda}^0{}_1(i\pi), \tag{41}$$

for Majorana spinors we obtain

$$\hat{\pi}\hat{\tau}\psi_1(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_1(-x), \qquad \hat{\pi}\hat{\tau}\psi_2(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_2(-x). \tag{42}$$

Thus up to an overall complex phase, quite remarkably we recognize (Mannheim JPA 2018) this transformation as acting as none other than the **linear** part of a CPT transformation since

$$\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)].$$

Note that for fermions $\hat{\pi}\hat{\tau}$ does not implement a PT transformation, but a CPT transformation, a thus first principles reason for charge conjugation.Thus the linear part of a CPT transformation is naturally associated with the complex Lorentz group.

14 PROOF OF THE CPT THEOREM WITHOUT HERMITICITY

But what about the antilinear part of a CPT transformation. That is fixed by probability conservation. Thus if impose just **INVARIANCE UNDER COMPLEX LORENTZ TRANSFORMATIONS** and **PROBABILITY CONSERVATION** one can extend the CPT Theorem (Mannheim JPA 2018) to non-Hermitian systems.

Then when C is separately conserved (as is the case below the threshold for particle production) CPT defaults to PT, to thus put PT symmetry on a quite secure theoretical footing.

15 SOLVING THE GRAVITY GHOST PROBLEM

As noted before, the nonrenormalizable Einstein gravity (propagator behaves as $1/k^2$) is based on the second-order Einstein-Hilbert action:

$$I_{\rm EH}=\frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}R^\alpha{}_\alpha, \tag{43}$$

while the renormalizable conformal gravity (propagator behaves as $1/k^4$) is based on the fourth-order derivative

$$I_{\rm W}=-2\alpha_g\int d^4x\,(-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa}-\tfrac{1}{3}(R^\alpha{}_\alpha)^2\right]. \tag{44}$$

The combination of second-order and fourth-order actions $I_{\rm EH}+I_{\rm W}$ would be of the same form as the scalar action

$$I_S\;=\;\tfrac{1}{2}\int d^4x\left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi-(M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi+M_1^2M_2^2\phi^2\right], \tag{45}$$

and thus lead to an associated propagator of the Pauli-Villars form

$$D(k)=\frac{1}{(k^2-M_1^2)(k^2-M_2^2)}=\frac{1}{(M_1^2-M_2^2)}\left(\frac{1}{k^2-M_1^2}-\frac{1}{k^2-M_2^2}\right). \tag{46}$$

Thus as with Pauli-Villars this theory is a unitary PT theory, and there are no ghosts after all. So fully viable.

But what about pure fourth-order. Setting $M_1^2=M_2^2$ is singular, so pure fourth-order theory is a non-diagonalizable CPT symmetric theory, and thus PT symmetric theory since $g_{\mu\nu}$ is C even. So also fully viable.

16 BUT WHY SHOULD ANYONE WHO DOES NOT LIKE HIGHER-DERIVATIVE GRAVITY CARE

Consider the Dirac action for a massless fermion coupled to a background geometry of the form

$$I_{\rm D} = \int d^4x(-g)^{1/2} i\bar{\psi}\gamma^c V^\mu_c(\partial_\mu + \Gamma_\mu)\psi, \qquad (47)$$

where the V^μ_a are vierbeins and $\Gamma_\mu = -(1/8)[\gamma_a, \gamma_b](V^b_\nu \partial_\mu V^{a\nu} + V^b_\lambda \Gamma^\lambda_{\nu\mu} V^{a\nu})$ is the spin connection that enables $I_{\rm D}$ to be locally Lorentz invariant.

As constructed, Γ_μ also enables $I_{\rm D}$ to be locally conformal invariant under

$$V^\mu_a \rightarrow e^{-\alpha(x)} V^\mu_a(x), \quad \psi(x) \rightarrow e^{-3\alpha(x)/2}\psi(x), \quad g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x).$$

We thus get local conformal invariance for free. In fact, other than the double-well potential, the entire $SU(3) \times SU(2) \times U(1)$ standard model is conformal invariant. Thus if fermion masses are generated dynamically (Mannheim PPNP 2017), then the entire standard model would be locally conformal invariant.

We now introduce the path integral $\int D[\psi]D[\bar{\psi}]\exp iI_{\rm D} = \exp(iI_{\rm EFF})$, and on performing the path integration on ψ and $\bar{\psi}$ obtain an effective action with leading term (t'Hooft 2010)

$$I_{\rm EFF} = \int d^4x(-g)^{1/2}C\left[R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}(R^{\alpha}{}_{\alpha})^2\right], \tag{48}$$

where C is a log divergent constant. Thus in the standard model itself we generate the conformal gravity action. But the standard model is ghost free. Since the fermion path integral is equivalent to a one loop Feynman diagram and since one cannot change the signature of a Hilbert space in perturbation theory, conformal gravity must be ghost free too. Because if it were not then the standard model would not be unitary.

Radiative corrections to $SU(3)\times SU(2)\times U(1)$ in an external gravitational field generate conformal gravity whether you like it or not, and thus generate a gravity theory that has a PT structure, again whether you like it or not. Gravity thus forces PT symmetry on the standard model.

17 SUMMARY

PT symmetry is quite ubiquitous in physics, and cannot be avoided or ignored, especially in the standard model. To establish it or CPT symmetry only requires complex Lorentz invariance and probability conservation. And if the theory of quantum gravity turns out to be conformal gravity, then one of the four fundamental forces in nature would be a PT theory.