Optimal Control Applied to Stellar Models in Relativistic Dynamics

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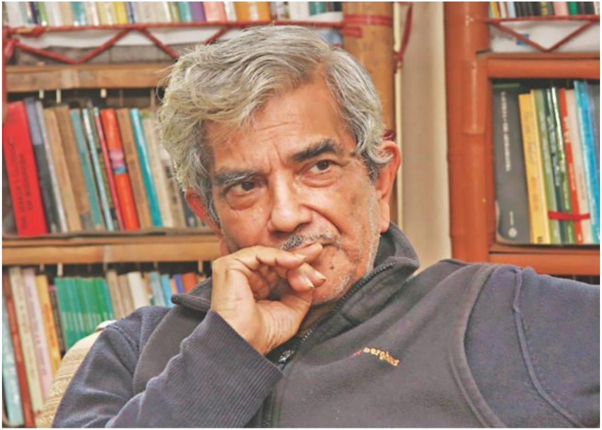
To the Memory of Renowned Cosmologist J. N. Islam



24 February 1939 - 16 March 2013

Presentation Summary

- Background
- Motivational Applications
- Theoretical Developments
- Application in Stellar Models
  - Nonlinear Stellar Model
  - Spherically Symmetric Stellar Equation
  - TOV Equation: Hydrostatic Equilibrium
  - Optimal Control Model
- Conclusions

Two Parts of This Talk

This talk consists of TWO parts:

- Basic Difference Between Optimization Problems and Optimal Control Problems
- Application of Optimal Control in Stellar Models

Standard Form Optimization Problems

The most general form of Optimization (NLP) problem:

$$(P_1) \quad \begin{cases} \text{Max/Min } \; f(x) \\ \text{subject to} \\ \qquad g(x) \geq (\leq)\; 0 \\ \qquad h(x) = 0 \\ \qquad x \in X \end{cases}$$

where $X \subset \mathbb{R}^n$ is a bounded set, x is a vector of n components and $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^n$ and $h : X \to \mathbb{R}^n$ are defined on X.

- The function f is usually called the *objective function* or *criterion function*.
- Each of the constraints $g_i(x) \leq 0, \;\; i = 1, \cdots, n$ is called an *inequality constraint*.
- Each of the constraints $h_i(x) = 0, \;\; i = 1, \cdots, n$ is called an *equality constraint*.

Solution of Optimization Problems

- A vector $x \in X$ satisfying all the constraints is called a *feasible solution* to the problem.
- The collection of all such points forms the *feasible region*.
- A feasible point x^* such that $f(x) \leq f(x^*)$ for each feasible point x is called *optimal solution*.

Theorem

(Weierstrass' Theorem): Let X be a nonempty, compact set, and let $f : X \to \mathbb{R}$ be continuous on X. Then, the problem Minimize$\{f(x) : x \in X\}$ attains its minimum, that is, there exists a minimizing solution to this problem.

Theorem

Let x^* be a local minimum of a convex optimization problem. Then, x^* is also a global minimum.

Example 1:

Consider the problem

$$(P_2) \quad \begin{cases} \text{Minimize } f \\ \text{subject to} \\ \qquad g_1(x,y) \leq 0 \\ \qquad g_2(x,y) \leq 0 \\ \qquad g_3(x,y) \leq 0 \end{cases}$$

where,

$f(x,y) = (x-3)^2 + (y-2)^2$

$g_1(x,y) = x^2 - y - 3$

$g_2(x,y) = y - 1$

$g_3(x,y) = -x$

Hence, the optimal solution occurs at the point $(2,1)$ and has an objective value equal to 2.

Solution of Problem (P_2)

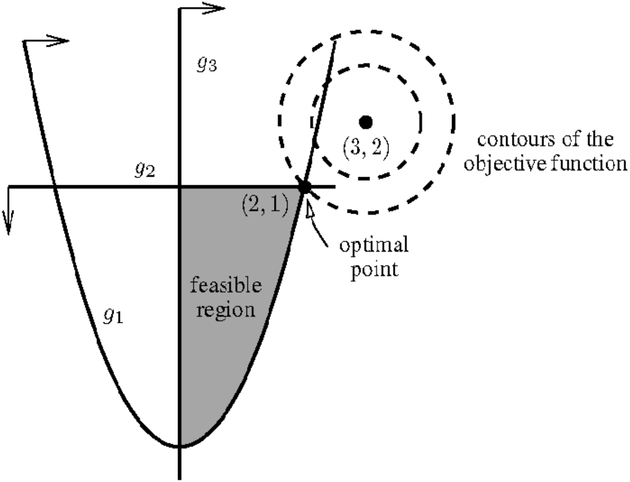


Figure 1: Geometric solution of a nonlinear problem P_2

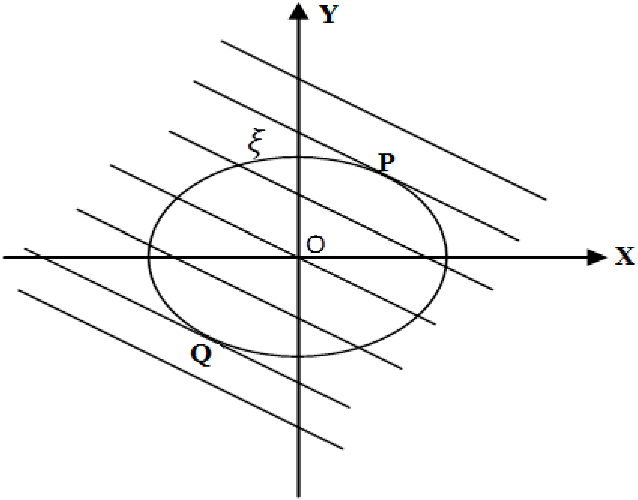
Role of Constraints on Optimization Problem I

Example 2:
Let us consider the function $f(x,y) = x + 2y$ such that $(x,y) \in \mathbb{R}$.
Then the function $f(x,y) =$ constant has neither a maximum nor minimum for all $(x,y) \in \mathbb{R}$.
That is, this optimization problem has no solution.
Now, we consider the problem

$$(P_3) \quad \begin{cases} \text{Minimize } f(x,y) \\ \text{subject to} \\ \ x^2 + 4y^2 = 8 \\ \ (x,y) \in \mathbb{R}. \end{cases}$$

Then the function $f(x,y) = \xi$ (say) satisfying the constraint $x^2 + 4y^2 = 8$ attains its maximum at $P(2,1)$ and minimum at $Q(-2,-1)$ on ξ.

Role of Constraints on Optimization Problem II



History of Optimal Control

- Calculus of Variations.
- Brachistochrone problem: *path of least time.*
- Newton, Leibniz, Bernoulli brothers, Jacobi, Bolza.

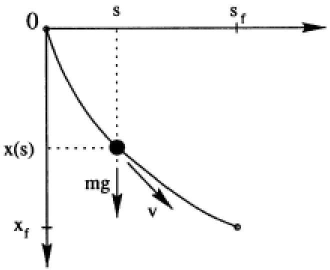


Figure 2: The Brachistochrone problem

- Pontryagin et al.(1958): Maximum Principle.
- Francis Clarke (1973): Nonsmooth Optimal Control.

Problem of Calculus of Variations

- The Basic Problem in the Calculus of Variations is that of finding an arc x^* which minimizes the value of an integral functional

$$J(x) = \int_0^T L(t, x(t), \dot{x}(t))\, dt$$

- over some class of arcs satisfying the boundary condition $x(0) = x_0$ and $x(T) = x_1$.

Here $[0,T]$ is a fixed interval, $L\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given function, and x_0 and x_1 are given points in $\mathbb{R}^n$.

Some Applications of Optimal Control

- Aerospace Engineering
- Robotic Engineering
- Mathematical Physics: Relativity and Cosmology

Figure 3: The diverse applications of optimal control

What is Dynamical System?

- A dynamic system which evolves over time, is described by state equation:
$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0$$
where $x(t)$ is state variable, $u(t)$ is control variable.
- The control aims to maximize/minimize the objective function:
$$J = l(x(0), x(T)) + \int_0^T L(t, x(t), u(t))dt$$
- Usually the control variable $u(t)$ will be constrained as follows:
$$u(t) \in U(t) \quad \text{a.e.} \;\; t \in [0, T].$$

Physical Constraints

Sometimes, we consider the following constraints:

- (1) Inequality constraint

$$g(t,x(t),u(t)) \leq 0, \quad \text{a.e. } t \in [0,T]$$

- (2) Constraints involving only state variables

$$h(t,x(t)) \leq 0, \quad \text{a.e. } t \in [0,T]$$

- (3) Terminal state

$$x(T) \in E \subset X(T)$$

where $X(T)$ is reachable set of the state variables at time T

Problem Statement

$$(P_B)\quad \begin{cases} \text{Minimize } J = l(x(0),x(T)) + \int_0^T L(t,x(t),u(t))dt \\ \text{subject to} \\ \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [0,T] \\ u(t) \in U(t) \quad \text{a.e. } t \in [0,T] \\ (x(0),x(T)) \in E. \end{cases}$$

- $[0,T]$ is a fixed interval.
- The function $f\colon [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes the system dynamics
- $U\colon [0,1] \to \mathbb{R}^m$ is a multifunction.
- $E \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set
- the scalars $l\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $L\colon [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ specify the endpoint constraints and cost.

Cost Functional

Definition

The objective functional

$$J = l(x(0), x(T)) + \int_0^T L(t, x(t), u(t))dt \tag{1}$$

to be minimized is called the *performance index* or *payoff* or *cost function*.

- $l(x(0), x(T))$ is called the terminal (endpoint) cost or salvage cost.
- the integral cost $\int_0^T L(t, x(t), u(t))dt$ is called the running cost or instantaneous cost.

Different Forms of OCPs

Optimal control problems are of three kinds: Bolza, Mayer and Lagrange forms.

Definition

The fairly general functional form with running and terminal costs is called the *Bolza* form of the objective functional. Problem (P_B) is a Bolza type optimal control problem.

Definition

When the Lebesgue integrable function $L\colon [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is absent from the cost functional (1) (i.e. $L \equiv 0$) and all others constraints remain the same, we obtain the *Mayer form* with cost $J = l(x(a), x(b))$.

Different Forms of OCPs (contd.)

Definition

If the function $l\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is absent from the cost functional (1) and all others data remain the same, we obtain the optimal control problem in *Lagrange form*; the cost is simply $J = \int_0^T L(t,x(t),u(t))dt$.

Remarks:

- Problem (P_B) is fixed time OCP (as interval $[0,T]$ is fixed)
- Free time OCP
- Minimum time OCP
- Constrained problems: state constrained or mixed constrained or both
- $\cdots$

State Augmentation

we can reformulate Bolza form (1) into Mayer form by means of the process called *state augmentation*. Let us take a new state variable y and define,

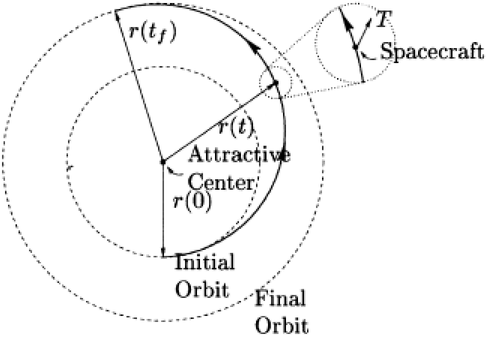
$$\begin{aligned} \dot{y} &= L(t, x(t), u(t)) \quad \text{a.e.} \\ y(0) &= 0. \end{aligned} \tag{2}$$

Then the problem (P_B) can be rewritten as following

$$(P_M) \quad \begin{cases} \text{Minimize } J = l(x(0), x(T)) + y(T) \\ \text{subject to} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, T] \\ \dot{y}(t) = L(t, x(t), u(t)) \quad \text{a.e. } t \in [0, T] \\ u(t) \in U(t) \quad \text{a.e. } t \in [0, T] \\ ((x(0), x(T)), y(0)) \in E \times \{0\}. \end{cases}$$

(P_M) is now in Mayer form.

The Maximal Orbit Transfer Problem



Orbit Transfer Model of Space Vehicle

The motion of the vehicle is governed by the rocket thrust and by the rocket thrust orientation, both of which can vary with time

$$\begin{cases} \text{Minimize } \; -r(t_f) \\ \text{over radial and tangential components of the thrust history,} \\ \qquad (T_r(t), T_t(t)),\ 0 \le t \le t_f, \text{ satisfying} \\ \dot{r}(t) = u, \\ \dot{u}(t) = v^2(t)/r(t) - \mu/r^2(t) + T_r(t)/m(t), \\ \dot{v}(t) = -u(t)v(t)/r(t) + T_t(t)/m(t), \\ \dot{m}(t) = -(\gamma_{\max}/T_{max})(T_r^2(t) + T_t^2(t))^{1/2}, \\ (T_r^2(t) + T_t^2(t))^{1/2} \le T_{\max}, \\ m(0) = m_0,\ r(0) = r_0,\ u(0) = 0,\ v(0) = \sqrt{\mu/r_0}, \\ u(t_f) = 0,\ v(t_f) = \sqrt{\mu/r(t_f)}. \end{cases}$$

Variables and Constants of the Model

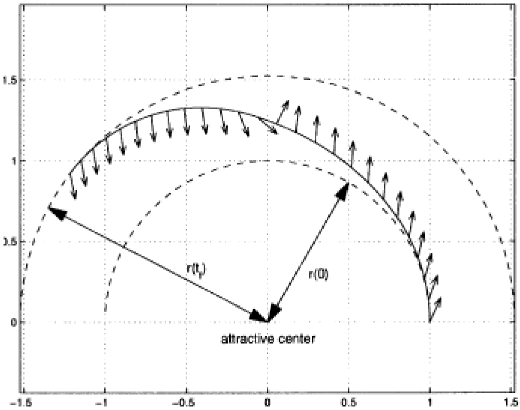
- The variables involved in the model are

$$\begin{aligned} r &= \text{radial distance of vehicle from attracting center,} \\ u &= \text{radial component of velocity,} \\ v &= \text{tangential component of velocity,} \\ m &= \text{mass of vehicle,} \\ T_r &= \text{radial component of thrust, and} \\ T_t &= \text{tangential component of thrust.} \end{aligned}$$

- The parameters and constants used in the model are

$$\begin{aligned} r_0 &= \text{initial radial distance,} \\ m_0 &= \text{initial mass of vehicle,} \\ \gamma_{\max} &= \text{maximum fuel consumption rate,} \\ T_{\max} &= \text{maximum thrust,} \\ \mu &= \text{gravitational constant of attracting center, and} \\ t_f &= \text{duration of maneuver.} \end{aligned}$$

An Orbit Transfer Strategy



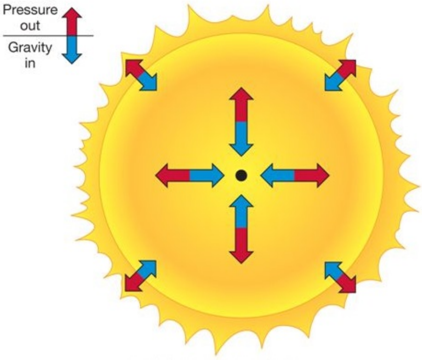
Hydrostatic Equiibrium/Balance

- A fluid is said to be in hydrostatic equilibrium or hydrostatic balance when it is at rest, or when the flow velocity at each point is constant over time.
- This occurs when external forces such as gravity are balanced by a pressure gradient force.
- The pressure-gradient force prevents gravity from collapsing Earth's atmosphere into a thin, dense shell,
- The gravity prevents the pressure gradient force from diffusing the atmosphere into space.
- Hydrostatic equilibrium is the current distinguishing criterion between *dwarf planets* and small S*olar System bodies*, and has other roles in *astrophysics* and *planetary geology*.

Schematic Diagram of Hydrostatic Equilibrium

Fusion keeps stars from collapsing under their own weight. Pressure from the outflowing hot gas balances the pressure of gravity.

This process is called **hydrostatic equilibrium**



Standard Form of Stellar Model

- One of the important applications of general relativity is the study of stellar models: their construction, equilibrium and stability.
- The standard form of general static isotropic metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right),$$

where the functions of gravitational fields $A(r)$ and $B(r)$ are to be determined by solving the Einstein's field equation in empty space

$$R_{ab} = 0.$$

- Here $A(r)$ and $B(r)$ can be calculated as

$$B(r) = \left(1 - \frac{2GM(r)}{r}\right) \text{ and } A(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1},$$

Equation of Stellar Model

- The simplest model is an isolated static sphere of perfect fluid. The vacuum outside the star has the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM(r)}{r}\right)dt^2 - \left(1 - \frac{2GM(r)}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

where M is the total mass-energy of the star.

- In the interior of the star, the spacetime is described by the static spherically symmetric metric

$$ds^2 = e^{2\nu} dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right),$$

where ν and m are functions of the coordinate radius r.

Differential Equations for Stellar Structure

- Energy-momentum tensor for a perfect fluid

$$T^{ab} = \left(\mu + p\right) U^a U^b - p g^{ab},$$

where μ and p are respectively, the total proper density and proper pressure, and U^a is the four-velocity of the matter such that,

$$g^{ab} U_a U_b = -1.$$

- The first law of thermodynamics relates the number density of baryons n to μ and p by the equation

$$\frac{dn}{d\mu} = \frac{n}{p+\mu}, \tag{3}$$

Fundamental Equation of Newtonian Astrophysics

- The fundamental equation of Newtonian astrophysics with relativistic corrections

$$-r^2 p'(r) = GM(r)\mu(r)\left(1+\frac{p(r)}{\mu(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{M(r)}\right)\left(1-\frac{2GM(r)}{r}\right)^{-1}. \tag{4}$$

- $p(r)$ is regarded as a function of $\mu(r)$ alone, with no explicit dependence on r

$$M'(r) = 4\pi r^2 \mu(r), \quad \text{with initial condition} \quad M(0) = 0. \tag{5}$$

where

$$M(r) = \int_0^r 4\pi r^2 \mu(r) dr.$$

Outside the Stars

- Outside the star, $p(r)$ and $\mu(r)$ vanish, and $M(r)$ is the constant $M(R)$.
- The constant $M(R)$ that appears in the asymptotic gravitational field $B(r) = A^{-1}(r) = \left(1 - \frac{2GM(r)}{r}\right)$ for $r \geq R$ must equal the mass M of the star, that is

$$M = M(R) = \int_0^R 4\pi r^2 \mu(r) dr.$$

Stability of the Stars I

- The solutions of the fundamental equations (4) and (5) represents an equilibrium state of the star.
- But it may be a state of stable or of unstable equilibrium.
- Our concern is only about the stable equilibrium.

Theorem

Stability of the Stars II

A star, consisting of a perfect fluid with constant chemical composition and entropy per nucleon, can only pass from stability to instability with respect to some particular radial normal mode, at a value of the central density $\mu(0)$ for which the equilibrium energy E and nucleon number Z are stationary, that is,

$$\frac{\partial E(\mu(0); s, \cdots)}{\partial \mu(0)} = 0, \text{ and } \frac{\partial N(\mu(0); s, \cdots)}{\partial \mu(0)} = 0.$$

By a radial normal mode is meant a mode of oscillation in which the density perturbation $\delta\mu$ is a function of r and t alone, and in which nuclear reactions, viscosity, heat condition, and radiative energy transfer play no role.

Tolman-Oppenheimer-Volkoff (TOV) equation

Theorem

Among all momentarily static and spherically symmetric configurations of cold, catalyzed matter which contain a specific number of baryons inside a sphere of radius R,

$$Z = \int_0^R 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} n(r)dr,$$

that configuration which extremizes the mass as sensed from outside,

$$M = m(r) = \int_0^R 4\pi r^2 \mu(r)dr,$$

satisfies the TOV equation of hydrostatic equilibrium,

$$\dot{p} = \frac{(p+\mu)(m+4\pi r^3 p)}{r(r-2m)}.$$

Relation of TOV Equation with Stellar Model

- Solutions of Einstein equations that are spherically symmetric and extremize the entropy of a perfect fluid for fixed total mass satisfy the Tolman- Oppenheimer-Volkoff (TOV) equation.
- The TOV equation is a general relativistic version of the well-known equation for hydrostatic equilibrium in a fluid with Newtonian gravity and has been extensively used in the study of relativistic stars.
- The TOV equation and the equation of effective mass $m(r)$ inside a sphere of surface area $4\pi r^2$ are equivalent to Bondi's equation

$$\frac{dr}{r} = \frac{du}{v/(\gamma-1)-u} \tag{6}$$

where $u(r) = m(r)/r$ and $v(r) = 4\pi r^2 p(r)$.

Stellar Model in Optimal Control Problem

- In this optimal control model, consider $\mu(r)$ as the control function,
- The number density of baryons is then a function of the control $n := n(\mu)$,
- A state function defining the system is $m(r)$,
- The constraint of fixed number of baryons Z is imposed by introducing another state function z,

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi r^2 \mu(r) \\ \frac{dz(r)}{dr} &= 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} n(\mu) \end{aligned} \tag{7}$$

with the initial conditions

$$m(0) = 0, \text{ and } z(0) = 0,$$

Model in Optimal Control Problem

Our aim is to determine the control function $\mu(r)$ which minimizes the objective function

$$M = m(R) = \int_0^R 4\pi r^2 \mu(r) dr,$$

subject to the dynamic constraints

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi r^2 \mu(r) \\ \frac{dz(r)}{dr} &= 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} n(\mu) \end{aligned} \tag{8}$$

with the initial conditions

$$m(0) = 0, \quad \text{and} \quad z(0) = 0,$$

and the terminal conditions

$$z(R) = Z$$

Optimal Control Problem (OCP)

$$
(\mathrm{P}) \begin{cases} \text{Minimize } m(R) \\ \text{subject to} \\ \quad \dot{x}(r) = f(x(r), u(r)) \;\; \text{for a.e. } r, \\ \quad u(r) \in U \;\; \text{for a.e. } r, \\ \quad x(0) = x_0 \end{cases}
$$

where

$$
x(r) = (m(r), z(r)), \;\; m(R) = \int_0^R 4\pi r^2 \mu(r) dr,
$$

$$
f(x) = \left(4\pi r^2 \mu(r), 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} n(\mu)\right),
$$

and the set U is the set of admissible controls such that

$$
u(r) = \mu(r) \in U.
$$

Optimality Condition in terms of Hamiltonian

- The Hamiltonian for the problem (P) takes the form

$$H(m,z,\mu,\lambda_1,\lambda_2) = \lambda_1 4\pi r^2 \mu(r) + \lambda_2 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} n(\mu),$$

where $\lambda = (\lambda_1, \lambda_2)$ denotes the vector of costate functions.

- Optimality condition is given by

$$\frac{\partial H}{\partial \mu}\Big|_{\mu=\mu^*} = 0$$

$$\Longrightarrow \lambda_1 4\pi r^2 + \lambda_2 4\pi r^2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} \frac{dn}{d\mu} = 0$$

$$\Longrightarrow \lambda_1 + \lambda_2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} \frac{dn}{d\mu} = 0$$

Adjoint Equations in terms of Hamiltonian

- The adjoint equations are given by

$$\lambda' = -\frac{\partial H}{\partial x}, \text{ here } \lambda = (\lambda_1, \lambda_2), \ x = (m, z)$$

which gives

$$\lambda_1' = -4\pi r \left(1 - \frac{2m(r)}{r}\right)^{-\frac{3}{2}} \lambda_2 n \tag{9}$$

and

$$\lambda_2' = 0$$

- With the transversality conditions

$$\lambda_1(R) = 1, \text{ and } \lambda_2(R) = \nu_2.$$

Condition of Hydrostatic Equilibrium

- Integrating $\lambda_2' = 0$ and using the transversality condition $\lambda_2(R) = \nu_2$, we get the costate function

$$\lambda_2 = \nu_2 = \text{constant}.$$

- Using $\dfrac{dn}{d\mu} = \dfrac{n}{p+\mu}$ to the optimality condition

$$\lambda_1 + \lambda_2 \left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} \frac{dn}{d\mu} = 0,$$

we get

$$\lambda_1 = -\frac{n}{p+\mu}\left(1 - \frac{2m(r)}{r}\right)^{-\frac{1}{2}} \lambda_2. \tag{10}$$

Condition of Hydrostatic Equilibrium (Contd.)

- Differentiating (10) and using (3) and (8), we get

$$\lambda_1' = -\left[p'\frac{r(r-2m)}{p+\mu} - 4\pi r^3\mu + m\right]\frac{n}{(p+\mu)r^2}\left(1-\frac{2m}{r}\right)^{-\frac{3}{2}}\lambda_2. \tag{11}$$

Hydrostatic Equilibrium: TOV Equation

- A straightforward comparison of (11) and (9), gives the condition of Hydrostatic Equilibrium of Stellar Model as

$$\dot{p} = \frac{(p+\mu)(m+4\pi r^3 p)}{r(r-2m)}. \tag{12}$$

Numerical Study of the Model

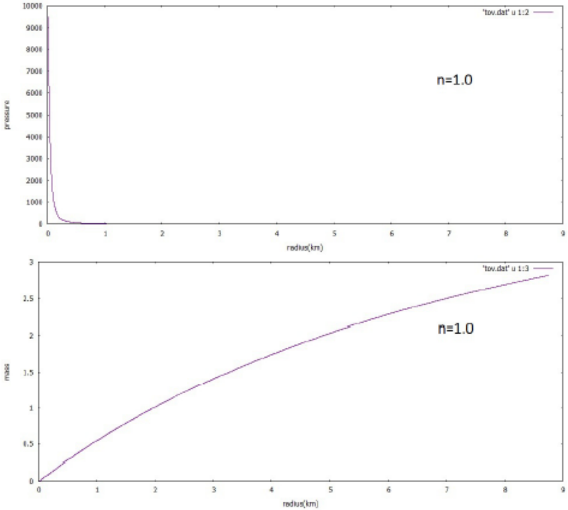


Figure 4: For $n = 1$ we got a near linear relation and in the TOV equation the mass is directly related to r^2

Numerical Study of the Model

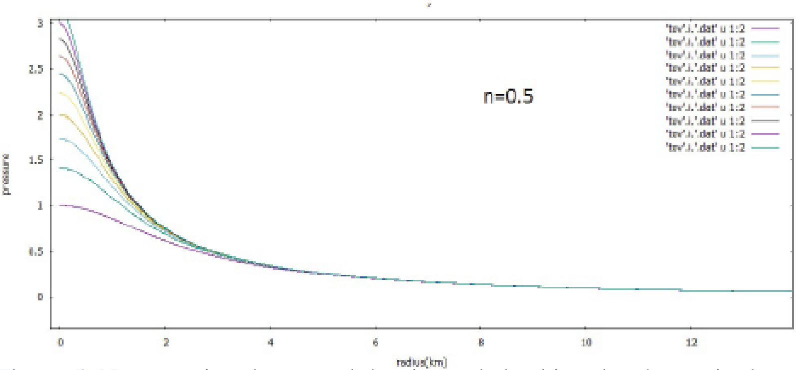


Figure 5: Now varying the central density and checking the change in the solution of the star for $n = 0.5$

A maximum mass at a finite value of the central density

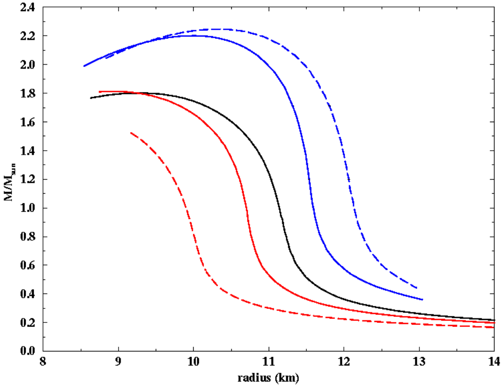


Figure 6: The general feature of these curves is the occurrence of a maximum mass at a finite value of the central density ($\rho_c \sim 10^{15} g/cm^3$)

Solutions of Bondi's equation for a fluid of massless quanta $\gamma = \frac{4}{3}$, $n = 4$.

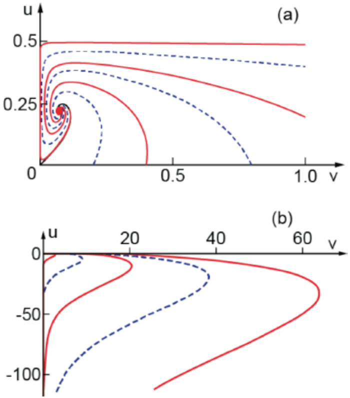


Figure 7: Solutions of Bondi's equation for a fluid of massless quanta $\gamma = \frac{4}{3}$, $n = 4$.

Conclusions

1. The condition of hydrostatic equilibrium of relativistic stellar models has been formulated as an optimal control problem.
2. A simple application of Pontryagins maximum principle has led directly to the TOV equation.
3. Numerical solution of TOV equation is presented.
4. Optimal Control can be applied to solve other mathematical problems in astrophysics, relativity and cosmology.

Bibliography I

[1] M. H. A. Biswas, M. A. K. Azad and M. A. Ansary, *Calculation of mass energy density for a collapsing dust sphere*, *Malaysian Journal of Mathematical Sciences*, **3**(2) (2009) 267–279.

[2] M. H. A. Biswas, *Necessary Conditions for Optimal Control Problems with State Constraints: Theory and Applications*, *PhD Thesis*, Department of Electrical and Computer Engineering, Faculty of Engineering, University of Porto, Portugal, 2013.

[3] M. H. A. Biswas, *Studies on Mathematics and Physics of Collapsing Stars*, *M. Phil. Thesis*, Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh, 2007.

Bibliography II

[4] M. H. A. Biswas, *Black Holes: The Ultimate Fate of the Collapsing Stars*, *Khulna University Studies*, **8**(1) (2007b) 183–150.

[5] S. Haggag and J. L. Safko, *The Condition of Hydrostatic Equilibrium of Stellar Models Using Optimal Control*, *Astrophysics and Space Science*, **283** (2003) 369–373.

[6] J. N. Islam, *The Ultimate Fate of the Universe*, First Edition, *Cambridge University Press*, UK, 1983.

[7] W. H. Zurek and Don N. Page, *Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation*, *Physical Review D*, **29**(4) (1984), 628–631.

Bibliography III

[8] S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity," *John Wiley and Sons, Inc., USA*, 1972.

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