4+1 Formalism for a Local Metric with Parameterized Evolution

Notes on the Slide Presentation

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Stueckelberg-Horwitz-Piron (SHP) Formalism in Relativity

The SHP framework emerged from the historical interaction between electrodynamics and relativity

- The symmetries of Maxwell's equations led Einstein to study the Lorentz group.
- Minkowski's tensor formulation of the Lorentz transformations led Fock to propose using proper time as the evolution parameter for a Newton-like force law producing the manifestly covariant formulation of electrodynamics.
- The study of particle/antiparticle pair processes in electrodynamics led Stueckelberg to introduce an external time τ and argue that the proper time of the motion must be τ-dependent in relativity theory.
- Removing the constraint in the 4D relativistic phase space led Horwitz and Piron to formulate covariant canonical theory of electromagnetic interactions.
- The covariant canonical theory of electromagnetic interactions led Horwitz, Saad, Arshansky to a 5D gauge theory of electrodynamics with 5 τ-dependent gauge potentials.
- The 5D gauge theory led to a picture of an evolving 4D block universe $\mathcal{M}(\tau)$.

Stueckelberg-Horwitz-Piron (SHP) Formalism

Covariant canonical mechanics with parameterized evolution

8D unconstrained phase space $\implies$ $\tau \neq$ proper time

- In SHP, the 8D phase space in unconstrained. In particular the mass shell constraint $\dot{x}^2 =$ constant is removed, and so the squared proper time $ds^2 = -\dot{x}^2 d\tau^2$ is a dynamical variable, not necessarily positive. This was the basis of Stueckelberg's argument against using proper time as as the evolution parameter.
- Moreover, any combination of phase space variables must have nonzero Poisson brackets with some other combination, including the Hamiltonian, making quantization impossible.
- Evolution parameter τ must be external to phase space

Canonical electrodynamics with scalar Hamiltonian ($K =$ total mass)

- The maximal gauge freedom for the classical Lagrangian $L = \frac{1}{2}\dot{x}^2$ leads to 5 τ-dependent potentials: $a_\alpha(x,\tau) \qquad \alpha,\beta,\gamma,\delta = 0,1,2,3,5 \qquad x^5 = c_5\tau.$
- The gauge-invariant Lagrangian can be written $L = \dfrac{1}{2}Mg_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + e\dot{x}^\alpha a_\alpha(x,\tau)$, suggesting a formal 5D symmetry that must be broken by matter.
- Objects with indices $\mu,\nu = 0,1,2,3$ belong to tensor representations of O(3,1), while the index 5 belongs to a scalar representation of O(3,1).
- Classical SHP electrodynamics is a system of events interacting through the 5 τ-dependent potentials, representing an integrable microscopic dynamics that reduces to Maxwell theory in an equilibrium limit.

Extend SHP to include pseudo-5D metric describing τ-evolution of spacetime

- We extend the SHP approach into general relativity by introducing a τ-dependent local metric leading to the Lagrangian $L = \dfrac{1}{2}Mg_{\alpha\beta}(x,\tau)\dot{x}^\alpha\dot{x}^\beta$ and examining the expected 5D to 4D+1 symmetry breaking

SHP — Geometry and Evolution

4D block universe $\mathcal{M}(\tau)$ at each τ

Physical event $x^\mu(\tau)$ in SHP

- An event $x^\mu(\tau)$ in SHP is an irreversible occurrence **at** time τ with spacetime coordinates x^μ.
- The formalism implements two aspects of time as distinct physical quantities:
  Coordinate time $x^0 = ct$ describing the event location,
  Stueckelberg time τ describing the chronological order of event occurrence.
- No event can change a previously occurring event $\longrightarrow$ no grandfather paradox

Evolution

- In SHP, the 4D Block universe $\mathcal{M}(\tau)$ **occurs** at τ, representing the 4D manifold of general relativity, comprising all of space and coordinate time x^0.
- The Hamiltonian K generates evolution of $\mathcal{M}(\tau)$ occuring at τ to an infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occuring at $\tau + d\tau$. The configuration of spacetime, including the past and future of $x^0 = ct$, may shift infinitesimally from chronological moment to moment in τ.
- Because K is an O(3,1) scalar and τ is an external parameter (Poincaré invariant by definition), there is no conflict with the general diffeomorphism invariance of general relativity.

Geometry and Trajectory

Standard approach to motion in general relativity

Two neighboring events in spacetime manifold $\mathcal{M}$ (instantaneous displacement)

- In the standard formulation of general relativity, neighboring points in spacetime coexist in a 4D block manifold $\mathcal{M}$.
- The metric $g_{\mu\nu}$ describes an invariant interval between two co-occuring points.
- The invariance of the interval is a *geometrical* statement about the manifold $\mathcal{M}$.

Trajectory — a curve in $\mathcal{M}$

- A trajectory is a curve defined by mapping some arbitrary parameter ζ to a set of events $x^{\mu}(\zeta)$, defined as a sequence by the parameterization.
- If the interval between any two events is timelike, we have

  $$-ds^2 = -dx^2 > 0 \qquad\qquad g = (-,+,+,+)$$

  and may take $d\zeta \rightarrow ds$ to parameterize the trajectory by proper time.

In 4D block universe $\mathcal{M}$

- In a clock universe, nothing actually moves or evolves.
- A trajectory is a sequence of displacements between co-occurring points in $\mathcal{M}$.
- We give an operational definition to the concept of "motion" by way of a trajectory with the property that if $s_1 \neq s_2$ then $x^0_1(s_1) \neq x^0_2(s_2)$, so that the parameterization by proper time assembles a set of points with changing coordinate time.

Path length $\longrightarrow$ Lagrangian

- Given a metric, one may define a Lagrangian for geodesic motion by minimizing path length.
- The square root form leads to the mass shell constraint, which can be removed by defining the Lagrangian for the squared path length.

Geometry and Evolution

Pseudo-5D metric

Two neighboring events:

- In the SHP approach we may consider the "distance" between events occurring at subsequent times, neighbors in sense of an association through the dynamical evolution of $\mathcal{M}(\tau) \to \mathcal{M}(\tau + \delta\tau)$.
- Consider events occurring at different locations of spacetime at subsequent times:
$$x^\mu(\tau) \in \mathcal{M}(\tau) \qquad\qquad \bar{x}^\mu(\tau + \delta\tau) \in \mathcal{M}(\tau + \delta\tau)$$

Distance

- The distance between these events is
$$dx^\mu = \bar{x}^\mu(\tau + \delta\tau) - x^\mu(\tau) \simeq \bar{x}^\mu(\tau) + \dot{\bar{x}}^\mu(\tau)\delta\tau - x^\mu(\tau) = \delta x^\mu + \dot{\bar{x}}^\mu \delta\tau$$
where the spacetime distance is $\delta x^\mu = \bar{x}^\mu(\tau) - x^\mu(\tau)$

Squared interval (referred to x coordinates)

- Using the formal notation $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5, \quad x^5 = c_5\tau$, we may define the squared interval by introducing a formal metric $g_{\alpha\beta}(x, \tau)$.

Contributions to interval

- The components $g_{\mu\nu}$ characterize the interval between two events of $\mathcal{M}$ at τ, and expresses the geometrical symmetries of spacetime manifold $\mathcal{M}$
- The component g_{55} characterizes the interval between one event in $\mathcal{M}(\tau)$ and another event in $\mathcal{M}(\tau + \delta\tau)$ and expresses the dynamical symmetries of evolution generated by Hamiltonian K

Example in space

Particle in 2D space — expanding disk with radius $R(\tau) = \frac{1}{2}g\tau^2$

Points on expanding disk

- We consider two subseqent points at the edge of an expanding disk in 2D space whose radius depends on τ.
- The expanding disk is not an inertial frame and there is no standard Lagrangian description of the system in this frame.

Distance

- By treating the evolving system as a manifold in pseudo-3D, we may consider a Lagrangian formed to minimze the path length.
- In polar coordinate the distance depends on θ and $R(\tau)$, with:
- Geometrical distances at fixed time: $\delta\theta = \bar{\theta} - \theta \qquad \delta R = \bar{R}(\tau) - R(\tau)$
- Dynamical distance determined by dynamical evolution:
  $$\delta_\tau R\left(\tau\right) = R\left(\tau + \delta\tau\right) - R\left(\tau\right) = g\tau\delta\tau$$

Interval

- The squared interval depends on intervals in R, θ, and τ

Pseudo-3D metric

- We express the squared interval in terms of pseudo-3D coordinates $\delta\zeta = (\delta R, \delta\theta, \delta\tau)$ and write a pseudo-3D metric for a dynamical problem in 2D

Example in space

Equations of motion

Lagrangian

- We can now write a Lagrangian describing minimal path length (geodesic motion) in the ζ coordinates.
- We write the Euler-Lagrange equations for R and θ, but since τ is non-dynamical ($\dot{\tau} \equiv 1$ by *a priori* constraint), there is no evolution equation for $\zeta_3 = \tau$.
- The Lagrangian is cyclic in θ, and so the θ equation expresses conservation of angular momentum
- The R equation includes the "fictitious" force $-Mg$ resulting from the dynamical expansion of the disk.

Qualitative result

- The particle at the edge of the disk sees a force $F = \ell^2/MR^3 - Mg$.
- For the case $Mg > \ell^2/MR^3$, the particle moves at the edge of the disk as if attracted by a gravitational force.
- The "fictitious" force $F_R = -Mg$ appears as "external" force that enters through evolution of the circular geometry.

Canonical Mechanics in General 5D Spacetime

Lagrangian

- We consider a true 5D spacetime manifold with an external time τ.
- The Lagrangian is the squared interval formed from the 5-velocity and the 5D metric is a function of the 5 coordinates.
- The action is therefore conserved under 5D transformations that contain O(3,1) as a subgroup, for example O(3,2) with metric signature $(-1, 1, 1, 1, -1)$ or O(4,1) with metric signature $(-1, 1, 1, 1, +1)$.

Euler-Lagrange $\longrightarrow$ geodesic equations

- The absolute derivative of velocity with respect to τ and the compatible connection vanishes.
- $\dot{x}^5$ is unconstrained and satifies a dynamic equation of motion.

Canonical momentum

- The canonical momentum is found in the usual way and the velocity may be obtained in terms of the momentum.

Conserved Hamiltonian

- The scalar Hamiltonian, representing the particle mass, is equal to the Lagrangian, which is conserved according to the equations of motion.

Poisson bracket

- The Poisson bracket of K with itself vanishes trivially. The Hamiltonian is seen to be conserved in this form because the metric is independent of the external time τ.

Break 5D symmetry $\longrightarrow$ 4D+1

Constrain non-dynamical scalar $x^5 \equiv c_5\tau$

- We require SHP general relativity to be 4D symmetric and break the 5D symmetry to 4D+1 by constraining $x^5 \equiv c_5\tau$.
- We thus replace $\dot{x}^5 \equiv c_5$ in the 5D Lagrangian.

Euler-Lagrange $\longrightarrow$ geodesic equations

- The equations of motion are the standard geodesic equations for $\mu = 0, 1, 2, 3$. By constraint $\ddot{x}^5 \equiv 0$ and so τ is non-dynamical.

Symmetry-broken connection

- The equations of motion are equivalent to 5D geodesic equations with broken symmetry.
- $\Gamma^\alpha_{\beta\gamma}$ take the form of the standard affine connection except that $\Gamma^5_{\alpha\beta}$ is constrained to vanish, regardless of any τ derivatives of $g_{\alpha\beta}$ or spacetime derivatives of g_{55}.
- Equivalently, by taking X^5 to be the scalar fifth component of an object $X^\alpha = (X^\mu, X^5)$, we require $D_\alpha X^5 = \partial_\alpha X^5$ with no connection term.

Hamiltonian

- Because $\dot{x}^5$ is not dynamical, there is no fifth momentum component p_5.
- Therefore the g_{55} term in the Lagrangian reverses sign in transformation to the Hamiltonian, which is no longer equal to the Lagrangian.
- The Hamiltonian is thus not conserved when the metric depends on τ.

Matter

Non-thermodynamic dust

- We define a spacetime event density, as the number of events per 4D volume.
- We define the event 5-current $j^\alpha(x, \tau)$ by combining the event density with the 5-velocity $(\dot{x}^\mu, c_5)$. The continuity equation (conservation of the 5-current) reflects the scalar nature of $\rho(x, \tau)$.
- The 5×5 mass-energy-momentum tensor contains the 4D energy momentum tensor and the 5-current density. $T^{\alpha\beta}$ is conserved by the continuity equation.

Einstein equations

- We pose field equations for $g_{\alpha\beta}$ by writing 5D indices in the Einstein equations.

Weak Field Approximation

Small perturbation to flat metric

- As in the standard approach to GR, we write a weak field approximation by introducing a small perturbation $h_{\alpha\beta}$ to the flat background metric.
- Therefore derivative of $g_{\alpha\beta}$ are derivatives of $h_{\alpha\beta}$ and we take $\left(h_{\alpha\beta}\right)^2 \approx 0$.

Define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \longrightarrow$ Einstein equations

- With this definition, and discarding terms in $\left(h_{\alpha\beta}\right)^2$, the Einstein equations take an approximate linear form.

Impose gauge condition $\partial_\lambda \bar{h}^{\alpha\lambda} = 0 \longrightarrow$ wave equation

- With this gauge condition the linearized equations decouple to a 5D wave equation for $\bar{h}_{\alpha\beta}$ sourced by $T_{\alpha\beta}$.

Solve with principal part Green's function

- This wave equation can be solved using the principal part Green's function known from SHP electrodynamics.

Post-Newtonian Approximation

Point source $X = (cT(\tau), \mathbf{0})$ in co-moving frame where $\dot{T} = 1 + \alpha(\tau)/2$ and $\alpha^2 \approx 0$

- As a post-Newtonian approximation we consider an event moving in its rest frame along the x^0-axis, with a slowly varying velocity.
- Since the particle mass is $m = M\left(\dot{x}^0\right)^2/c^2 = M\dot{t}^2$ this represents a particle with τ-varying mass.
- We take this variation to be small with $\dot{t} = \dot{T} = 1 + \alpha(\tau)/2$ and $\alpha^2 \approx 0$.
- This leads to a number density in which we smooth the t location along its axis with some distribution function ρ.

Writing $M(\tau) = m\,\rho\,(t - T(\tau))$ a slowly varying density function

- Using the Green's function solution to the wave equation we calculate the nonzero components of the perturbed metric and the connection.
- We neglect terms in c_5^2/c^2 which must be small from considerations in SHP electrodynamics.
- The nonzero components of the connection involve $\partial_\nu h_{00}$, as in the standard approach to GR, and $\partial_\tau h_{00}$ through the newly introduced τ-dependence.

Equations of motion for a test particle in spherical coordinates putting $\theta = \pi/2$

- From the connection we may write equations of motion for a test particle in the perturbed gravitational field produced by $g_{\alpha\beta}$.
- Transforming to spherical coordinates and using orbital symmetry to put $\theta = \pi/2$ the equations of motion split into equations for t, ϕ, and R.
- The t equation reduces to $\ddot{t} = 0 \to \dot{t} = 1$ for $\alpha(\tau) = 0$ recovering nonrelativistic motion.
- The ϕ equation leads to conservation of angular momentum.
- The R equation is equivalent to the nonrelativistic equation on the LHS, but is driven by a force on the RHS dependent on the variation of mass.

Post-Newtonian Evolution

Solution to t equation neglecting $\dot{R}/c \ll 1$ and $\partial_\tau \rho \approx 0$

- Assuming nonrelativistic velocity $\dot{R}/c \ll 1$ and a very slowly varying mass, the t equation can be solved exactly, giving velocity as an exponential of the perturbation.
- Taking $\alpha = 0$ recovers $\dot{t} = 1$ and the Newtonian case.

Using solution to t equation in radial equation

- Inserting the equation for $\dot{t}$ in the R equation leads to

$$\frac{d}{d\tau}\left\{\frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{L^2}{M^2R^2} - \frac{GM}{R}\left(1 + \frac{1}{2}\alpha\left(\tau\right)\right)\right\} = -\frac{GM}{2R}\frac{d}{d\tau}\alpha\left(\tau\right)$$

- We recognize the LHS as $\dfrac{dK}{d\tau}$ where K is particle Hamiltonian, equal to the dynamic particle mass associated with the motion of the test particle.
- On the RHS we find a driving force produced by the variation of mass in the source event.
- We obtain a picture of a source event of varying mass inducing a perturbation of the gravitational field that leads to geodesic motion of a test particle with varying mass.
- In other the source is transmitting mass across spacetime by way of the τ-evolving metric.
- The Hamiltonian is conserved when the mass variation in the source vanishes, and the R equation recovers the nonrelativistic Newtonian case.

3+1 Formalism in General Relativity

Time evolution formalism for Einstein equations

- The 3+1 formalism in general relativity reframes the field equations into an intial value problem. As a computational scheme, it solves the Einstein equations by decomposing spacetime into simultaneous (equal-t) spacelike submanifolds, and finding equations for the t-evolution from submanifold to submanifold.
- Beyond providing a useful framework in numerical relativity, the initial value problem emphasizes the background independence of GR — one seeks a description of spacetime as a trajectory of spaces, both the solution and the physical setting of classical physics.
- This decomposition (*foliation*) of spacetime breaks manifest covariance.
- The formalism can be extended to 4+1 SHP general relativity in a natural way, because the breaking of 5D $\longrightarrow$ 4D+1 symmetry merely makes explicit the decomposition of 5D pseudo-spacetime into tensor and scalar representations of O(3,1).
- The 3+1 formalism decomposes the 10 components of the spacetime metric into a 6 component space metric, a 3 component (spatial) shift vector, and a 1 component (time) lapse function.
- Projecting the 10 components of the Einstein equations onto a 3D spacelike hypersurface of the 4D spacetime, leads to a 6 component partial differential equation (PDE) of second order in ∂_t, and 4 constraints on involving the space metric, its 1^{st} t-derivative, and the energy-momentum tensor.

Based on mathematics of embedded hypersurfaces

- The 3+1 formalism is based on the work of Darmois (1927), Lichnerowicz (1939), Choquet-Bruhat (1952) who studied the relation of the intrinsic structure of hypersurfaces and the extrinsic structure imposed by embedding hypersurface in a larger manifold.
- A closely related method is that of Arnowitt, Deser, Misner (1962). In the ADM formulation of GR, the space metric and extrinsic curvature become conjugate canonical field variables in a Hamiltonian formulation.

4+1 formalism generalizes 3+1 framework as presented in:

- The work presented here generally follows and extends the exposition of the 3+3 formalism in the papers of Gourgoulhon (2007), Bertschinger (2005), ADM (1962), Isham (1992), Blau (2020)

Schematic Outline of 3+1 Formalism

As an introduction to the 4+1 formalism presented here, we begin with a schematic outline of the 3+1 formalism.

Define a foliation of 4D spacetime

- Choose a scalar field $t(x)$ on spacetime for $x \in \mathcal{M}$.
- We seek the hypersurfaces comprised of simultaneous spacetime points x sharing $t(x) = constant$.
- Thus we choose $t(x)$ to identify spacelike hypersurfaces (3D space manifold at time t).
- These hypersurfaces have the properties that (1) each vector V tangent to each hypersurface is spacelike ($V^2 > 0$) and (2) there exists a timelike ($n^2 = -1$) vector n normal to the hypersurface.

Projection operators split tangent and normal components of 4D objects

- We define a projection operator onto the spacelike hypersurface and a projection onto the timelike normal n.
- The induced space metric γ_{ij} is the spacelike projection of the spacetime metric $g_{\mu\nu}$.
- The spacetime metric $g_{\mu\nu}$ can be reexpressed in terms of the induced space metric γ_{ij}, the (normal) lapse N, and the (tangent) shift N^i.
- The spacelike projection of the 4D covariant derivative is compatible with γ_{ij}
- The spacelike projection of the 4D curvature tensor is the usual (intrinsic) curvature tensor in 3D.
- The timelike projection of the 4D curvature tensor is reexpressed as the extrinsic curvature of the t-evolving spacelike hypersurface.

Einstein equations

- We then project the Einstein equations onto the spacelike hypersurface and the timelike normal.
- The 10 components of Einstein equations split into two groups:

  6 second order PDEs describing the t-evolution of the space metric γ_{ij}

  4 non-evolving constraints on the initial conditions

4+1 Formalism in SHP General Relativity

Schematic Outline I: The Embedding

Here we present a schematic outline of the 4+1 formalism described in the following.

5D pseudo-spacetime coordinates $X = (x, c_5\tau) \in \mathcal{M}_5 = \mathcal{M} \times R$

- We define the manifold $\mathcal{M}_5$ as an admixture of 4D spacetime geometry and τ-evolution, comprised of points $X = (x, c_5\tau)$.
- In flat pseudo-spacetime the metric is $g_{\alpha\beta} \to \eta_{\alpha\beta} = \operatorname{diag}(-1,1,1,1,\sigma)$ where $\sigma = \pm 1$

Natural foliation of 5D pseudo-spacetime $\mathcal{M}_5 \longrightarrow$ spacetime hypersurface $\sum_\tau$

- From the construction of $\mathcal{M}_5$ the natural foliation is to define the time field as selecting the 5^{th} component of X^α. This defines the spacetime hypersurface and the 4D spacetime embedded in $\mathcal{M}_5$.
- From the time function we may construct a embedded spacetime hypersurface
$$\tau(X) \longrightarrow \sum_{\tau_0} = \left\{ X \in \mathcal{M}_5 \,\middle|\, S(X) = \tau(X) - \tau_0 = 0 \right\}$$
- The vector n that is normal to hypersurface $\sum_{\tau_0}$ is τ-like ($n^2 = \sigma = \pm 1$)

Projection operators split tangent and normal components of 5D objects

- We construct projection operators onto spacetime and onto the normal vector.
- The induced spacetime metric $\gamma_{\mu\nu}$ is the projection onto $\mathcal{M}$ of 5D metric $g_{\alpha\beta}$
- The 5D metric $g_{\alpha\beta}$ is expressed in terms of the spacetime metric $\gamma_{\mu\nu}$, the lapse N, and the (tangent) shift N^μ.
- The spacetime projection of the 5D covariant derivative is compatible with $\gamma_{\mu\nu}$.
- The spacetime projection of the 5D curvature becomes the usual 4D (intrinsic) curvature $R_{\mu\nu\lambda\rho}$.
- The τ projection of the 5D curvature becomes the extrinsic curvature of τ-evolution $K_{\alpha\beta}$.

Schematic Outline II: Einstein Equations

5D Einstein equations

- The Einstein equations are obtained by extending the indices to 5D.

Bianchi relation

- The 5 components of the Bianchi relation impose 5 constraints on solutions to the field equations.

Project Einstein equations onto spacetime hypersurface and τ-like normal

- The 5D Einstein equations are projected onto the spacetime hypersurface and the τ-like normal.
- The projects decompose the 15 components of the field equations into two groups: (1) 10 unconstrained second order PDEs describe the τ-evolution of the spacetime metric $\gamma_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$,
(2) 5 non-evolving constraints on the initial conditions: $\left\{\gamma_{\mu\nu}, \partial_\tau \gamma_{\mu\nu}, T_{\alpha\beta}\right\}$.

Foliation of 5D Pseudo-Spacetime $\mathcal{M}_5 = \mathcal{M} \times R$

Embedding

- The embedding Φ maps $\mathcal{M}$ into $\mathcal{M}_5$ by associating $x^\mu \in \mathcal{M},\ \mu = 0,1,2,3$ with $X^\alpha = (x, c_5\tau) \in \mathcal{M}_5,\ \alpha = 0,1,2,3,5$.

4D hypersurface — implicitly defined submanifold

- The time field $\tau(X)$ selects $\tau = X^5/c_5$ for any point $X^\alpha \in \mathcal{M}_5$.
- The 4D hypersurface $\sum_{\tau_0}$ comprises all points in $\mathcal{M}_5$ for which $\tau(X) = \tau_0$.
- The 4D manifold $\mathcal{M}$ is studied through the projection of objects in $\mathcal{M}_5$ onto the 4D submanifold $\sum_{\tau_0}$.

Rank 4 Jacobian

- The rank 4 Jacobian $E^\alpha_\mu = \left(\dfrac{\partial X^\alpha}{\partial x^\mu}\right)_{\tau_0}$ provides a basis for $\mathcal{T}_x\left(\sum_{\tau_0}\right)$, the tangent space of $\sum_{\tau_0}$.

Unit normal to $\sum_{\tau_0}$

- The unit normal $n_\alpha = \sigma \left|g^{55}\right|^{-1/2} \partial_\alpha S(X)$ is orthogonal to the basis E_μ and is a unit vector in the sense of $n^2 = \sigma$.

Induced metric on $\mathcal{M}$

- The induced metric on $\mathcal{M}$ is found by restricting displacements in $\mathcal{M}_5$ to displacements in $\sum_{\tau_0}$, so that $dX^\alpha = \dfrac{\partial X^\alpha}{\partial x^\mu} dx^\mu = E^\alpha_\mu dx^\mu$.

Projection Operators

Projections

- We require projection operators onto the normal and transverse components of the unit normal n.
- The transverse components are in $\mathcal{T}_x(\mathcal{M}_5)$, the tangent space of $\sum_\tau$.

Normal Projection Operator

- From $A_{\parallel} = \sigma(A \cdot n)\, n$ we write the normal projection as $N_{\alpha\beta} = \sigma n_\alpha n_\beta$.

Tangent Projection Operator

- From $A_{\perp} = A - \sigma(A \cdot n)\, n$ we write the tangent projection as $P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta$.

Completeness Relation

- We express the identity operator as the completeness relation $\delta^\alpha_\beta = P^\alpha_\beta + \sigma n^\alpha n_\beta$.

Projector $P_{\mu\nu}$ restricted to $\sum_\tau$ is metric $\gamma_{\mu\nu}$

- For any vector V in the tangent space of $\mathcal{M}_5$ the projection $V^\alpha_\perp = P^\alpha_\beta V^\beta$ is in the 4D tangent space of $\sum_\tau$.
- Thus, there is some $v \in \mathcal{T}_x(\mathcal{M})$ such that $V^\alpha_\perp = v^\mu E^\alpha_\mu$. This establishes the correspondence between $\mathcal{T}_x(\sum_\tau)$ and $\mathcal{T}_x(\mathcal{M})$.
- For any vector $V^\alpha \in \mathcal{T}_x(\sum_\tau)$ the action of $P_{\mu\nu}$ is identical to the action of the metric $\gamma_{\mu\nu}$.

Decomposition of the Metric

Time evolution

- The embedding $X^\alpha = (x, \tau)$ leads naturally to the hypersurface $\sum_{\tau_0} = \{X^\alpha \mid \tau(X) = \tau_0\}$ and the trajectory $X^\alpha(\tau) = \{X^\alpha(x, \tau) \mid x = x_0\}$.
- A general trajectory will have a normal component pointing out of the hypersurface to $\sum_{\tau_0+\delta\tau}$ and a tangent component in the hypersurface $\sum_{\tau_0}$.
- Using the unit normal n as a basis for the normal component and E^α_μ as a basis for the tangent component, an infinitesimal evolution $\delta\tau$ leads to
$$\delta X^\alpha = \left(Nn^\alpha + N^\mu E^\alpha_\mu\right)\delta\tau.$$
- The function N is called the Lapse and the vector N^μ is called the Shift.

Spacetime shift

- A shift in the spacetime position δx^μ at fixed τ, must be within the equal-τ hypersurface and so leads to $\delta X^\alpha = E^\alpha_\mu \delta x^\mu$.

5D interval

- Writing a general displacement as $dX^\alpha = Nn^\alpha c_5 d\tau + E^\alpha_\mu (N^\mu c_5 d\tau + dx^\mu)$ the path length for a trajectory in 5D can be decomposed into displacements $d\tau$ and dx^μ.

Decomposition of metric using $n^2 = \sigma \qquad n_\alpha E^\alpha_\mu = 0 \qquad \gamma_{\mu\nu} = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu$

- Expanding the expression for ds^2 we find
$$g_{\mu\nu} = \gamma_{\mu\nu}$$
$$g_{\mu 5} = N_\mu$$
$$\sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu$$
- The inverse is
$$g^{\mu\nu} = \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu$$
$$g^{\mu 5} = -\sigma \frac{1}{N^2} N^\mu$$
$$g^{55} = \sigma \frac{1}{N^2}$$

Intrinsic Geometry

Covariant derivatives — compatibility — curvature

- The intrinsic geometry of $\sum_\tau$ refers to the structure of the hyperspace as a manifold without regard to its embedding in $\mathcal{M}_5$.
- On $\mathcal{M}_5$ the metric $g_{\beta\gamma}$ leads to the unique compatible connection $\Gamma_{\sigma\beta\gamma}$ that guarantees $\nabla_\alpha g_{\beta\gamma} = 0$ for the covariant derivative and leads to the Riemann curvature tensor through the Ricci equation $\left[\nabla_\beta, \nabla_\alpha\right] X_\delta = X_\gamma R^{\gamma}_{\delta\alpha\beta}$.
- Similarly, on $\mathcal{M}$, $\quad \gamma_{\mu\nu} \longrightarrow \Gamma_{\mu\nu\lambda} \longrightarrow D_\mu \gamma_{\nu\lambda} = 0 \longrightarrow \left[D_\nu, D_\mu\right] X_\rho = X_\lambda R^{\lambda}_{\rho\mu\nu}$

Projected Covariant Derivative

- Decomposition of the 5D Einstein equations to 4D+1 requires decomposition of the covariant derivative and curvature by projection onto tangent and normal hypersurfaces.
- For a vector $V \in \mathcal{M}_5$ we define the projected covariant derivative as the projection onto $\sum_\tau$ of the covariant derivative of the projected part of V.
- Writing $D_\alpha = \bar{\nabla}_\alpha$ for the projected covariant derivative allows us to associate D_α with the covariant derivative on $\mathcal{M}$ through $D_\mu = E^{\alpha}_{\mu} D_\alpha$.

Projected Curvature $\bar{R}^{\gamma}_{\delta\alpha\beta}$

- When its operation is restricted to objects in the tangent space of $\sum_\tau$, we can write the metric $\gamma_{\beta\gamma} = g_{\beta\gamma}$.
- Thus, on $\sum_\tau$ the metric $\gamma_{\beta\gamma}$ leads to the unique connection $\bar{\Gamma}_{\sigma\beta\gamma}$ for which $D_\alpha \gamma_{\beta\gamma} = 0$, which in turn leads to the projected curvature tensor defined through $\left[D_\beta, D_\alpha\right] X_\delta = X_\gamma \bar{R}^{\gamma}_{\delta\alpha\beta}$
- The relationship between $\bar{R}^{\gamma}_{\delta\alpha\beta}$ on the 4D manifold $\sum_\tau$ and $R^{\lambda}_{\rho\mu\nu}$ on the 4D manifold $\mathcal{M}$ is required for the decomposition of the Einstein equations.

Extrinsic Geometry

Extrinsic Curvature

Curvature of $\mathcal{M}$ as manifold embedded in $\mathcal{M}_5$ as $\sum_\tau$

- Extrinsic curvature refers to the structure of $\mathcal{M}$ in relation to its embedding in $\mathcal{M}_5$ as the manifold $\sum_\tau$.
- As τ advances and the spacetime structure of $\mathcal{M}$ evolves, the unit normal n will evolve, evolving the structure of $\sum_\tau$.
- The gradient $\nabla_\gamma n_\delta$ is an object in $\mathcal{M}_5$.
- The extrinsic curvature is the projection of this gradient onto $\sum_\tau$ as $K_{\alpha\beta}$.
- Using $n^2 = \sigma$ we see that the gradient is already tangent to $\sum_\tau$ in the β index, and so the projection on this index is unnecessary (since a projection is idempotent).
- Using the explicit form of n_β and expanding the projector, one finds an expression for the extrinsic curvature as $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$.

Extrinsic Geometry

Evolution of Hypersurface $\sum_\tau$

Time Evolution of $\sum_\tau$

- Under an infinitesimal time displacement $\delta\tau$ a point X^α of $\mathcal{M}_5$ will evolve along $E_5^\alpha = (\partial_5)^\alpha = Nn^\alpha + N^\mu E_\mu^\alpha$
- Defining $m^\alpha = Nn^\alpha$ with $m^2 = \sigma N^2$ leads to the basis vector in the time direction in the form $\partial_5 = m + \mathbf{N}$
- Using $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$ we are led to an expression for the gradient of the vector m in terms of the extrinsic curvature:
$$\nabla_\beta m_\alpha = -NK_{\beta\alpha} - n_\beta D_\alpha N + n_\alpha \nabla_\beta N$$

Lie derivative of $\gamma_{\mu\nu}$ along ∂_5

- We characterize the time evolution of the spacetime metric as the Lie derivative along the time direction: $\mathcal{L}_5 = \mathcal{L}_m + \mathcal{L}_\mathbf{N}$.
- Using the definition of the Lie derivative and the expression relating the gradient and extrinsic curvature, we are led to the simple form $\mathcal{L}_m\,\gamma_{\alpha\beta} = -2NK_{\alpha\beta}$
- This allows us the write an evolution equation for the spacetime metric as:
$$\mathcal{L}_5\,\gamma_{\alpha\beta} - \mathcal{L}_\mathbf{N}\,\gamma_{\alpha\beta} = -2NK_{\alpha\beta} \longrightarrow \mathcal{L}_5\,\gamma_{\mu\nu} - \mathcal{L}_\mathbf{N}\,\gamma_{\mu\nu} = -2NK_{\mu\nu}$$

Decomposition of the Riemann Tensor

Gauss-Codazzi Relations

Using $\quad \delta^{\alpha'}_{\alpha} = P^{\alpha'}_{\alpha} + \sigma n_\alpha n^{\alpha'} \qquad E^{\alpha}_{\mu} P^{\alpha'}_{\alpha} = E^{\alpha'}_{\mu} \qquad n_\alpha E^{\alpha}_{\mu} = 0$ to expand

- The decomposition of the Riemann tensor is accomplished by operating on each index with the completeness relation, and expanding in terms of the tangent and normal projects.
- Taking account of the symmetries of $R^{\gamma}_{\delta\alpha\beta}$, $E^{\alpha}_{\mu} P^{\alpha'}_{\alpha} = E^{\alpha'}_{\mu}$, and $n_\alpha E^{\alpha}_{\mu} = 0$, the resulting terms are:

$$R^{\gamma}_{\delta\alpha\beta} = \delta^{\alpha'}_{\alpha}\,\delta^{\beta'}_{\beta}\,\delta^{\gamma}_{\gamma'}\,\delta^{\delta'}_{\delta}\,R^{\gamma'}_{\delta'\alpha'\beta'} \longrightarrow \begin{cases} E^{\alpha}_{\mu}\,E^{\beta}_{\nu}\,E^{\lambda}_{\gamma}\,E^{\delta}_{\sigma}\,P^{\alpha'}_{\alpha}\,P^{\beta'}_{\beta}\,P^{\gamma}_{\gamma'}\,P^{\delta'}_{\delta}\,R_{\delta'\alpha'\beta'\gamma'} = R^{\lambda}_{\sigma\mu\nu} \\ E^{\alpha}_{\mu}\,E^{\beta}_{\nu}\,E^{\lambda}_{\gamma}\,P^{\gamma}_{\gamma'} n^{\delta}\,P^{\alpha'}_{\alpha}\,P^{\beta'}_{\beta}\,R^{\gamma'}_{\delta\alpha'\beta'} = \sigma N\,R^{\lambda}_{5\mu\nu} \\ E^{\alpha\mu}\,E^{\beta}_{\nu}\,P_{\alpha\alpha'}\,n^{\delta}\,P^{\beta'}_{\beta}\,n^{\gamma}\,R^{\alpha'}_{\delta\beta'\gamma} = N^2\,R^{\mu}_{5\nu5} \end{cases}$$

Gauss relation

- Writing the projected Ricci equation $\left[D_\beta, D_\alpha\right] V_\delta = V_\gamma \bar{R}^{\gamma}_{\delta\alpha\beta}$ for V tangent to $\sum_\tau$ leads to the decomposition $R^{\mu}{}_{\nu\lambda\rho} = \bar{R}^{\mu}{}_{\nu\lambda\rho} - \sigma\left(K^{\mu}_{\lambda}K_{\rho\nu} - K^{\mu}_{\rho}K_{\lambda\nu}\right)$.
- This expression decomposes the curvature on $\mathcal{M}$ into the projected curvature and extrinsic curvature on $\sum_\tau$.

Codazzi relation

- Writing the Ricci equation $\left[\nabla_\beta, \nabla_\alpha\right] n^\gamma = R^{\gamma}_{\gamma'\alpha\beta} n^{\gamma'}$ for the unit normal n in $\mathcal{M}_5$ leads to the decomposition $R^{5}_{\mu\nu\lambda} = \sigma\dfrac{1}{N}\left(D_\lambda K_{\nu\mu} - D_\nu K_{\lambda\mu}\right)$.
- This expression decomposes the 5-component of the curvature on $\mathcal{M}_5$ into the lapse and extrinsic curvature on $\sum_\tau$.

Evolution of the Extrinsic Curvature

Decompose Ricci equation for unit normal n

- Projecting the Ricci equation for unit normal n once onto n and twice onto $\sum_\tau$ leads to

$$P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_{\beta} \left(\nabla_{\beta'} \nabla_{\gamma'} n^{\alpha'} - \nabla_{\gamma'} \nabla_{\beta'} n^{\alpha'} \right) = P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_{\beta} \, R^{\alpha'}_{\delta\beta'\gamma'} n^{\delta}$$

- and using $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$ we obtain

$$P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_{\beta} \, R^{\alpha'}_{\delta\beta'\gamma'} n^{\delta} = -K_{\alpha\gamma} K^{\gamma}{}_{\beta} + \frac{1}{N} D_\beta D_\alpha N + P^{\gamma}{}_{\alpha} P^{\delta}{}_{\beta} \, n^{\varepsilon} \nabla_\varepsilon K_{\gamma\delta}$$

Lie derivative of $K_{\alpha\beta}$

- Taking the Lie derivative of $K_{\alpha\beta}$ along m we have

$$\left(\mathcal{L}_5 - \mathcal{L}_{\mathbf{N}}\right) K_{\alpha\beta} = \mathcal{L}_m \, K_{\alpha\beta} = m^\gamma \nabla_\gamma K_{\alpha\beta} + K_{\gamma\beta} \nabla_\alpha m^\gamma + K_{\alpha\gamma} \nabla_\beta m^\gamma$$

  so that using $\nabla_\beta m^\alpha = -N K^\alpha_\beta - n_\beta D^\alpha N + n^\alpha \nabla_\beta N$ leads to

$$P^{\alpha'}_{\alpha} P^{\beta'}{}_{\beta} R_{\;\alpha'\beta'} = \sigma \frac{1}{N} \mathcal{L}_m \, K_{\alpha\beta} + \sigma \frac{1}{N} D_\alpha D_\beta N + \bar{R}_{\alpha\beta} - \sigma K K_{\alpha\beta} + \sigma 2 K^{\delta}_{\alpha} K_{\beta\delta}$$

- This expression decomposes the Ricci tensor on $\mathcal{M}_5$ into the lapse, extrinsic curvature and projected Ricci tensor on $\sum_\tau$.
- Through the Lie derivative, it provides an evolution equation for the extrinsic curvature.

Decomposition of the Einstein Equations

Evolution Equation for $K_{\alpha\beta}$

Einstein equations

- We rewrite the 5D field equations in the form $R_{\alpha\beta} = \dfrac{8\pi G}{c^4}\left(T_{\alpha\beta} - \dfrac{1}{2}g_{\alpha\beta}T\right)$ where $T = g^{\alpha\beta}T_{\alpha\beta}$.

Decompose $T_{\alpha\beta}$

- Using the completeness relation to decompose the mass-energy-momentum tensor as $T_{\alpha\beta} = T_{\alpha'\beta'}\left(P_\alpha^{\alpha'} + \sigma n^{\alpha'}n_\alpha\right)\left(P_\beta^{\beta'} + \sigma n^{\beta'}n_\beta\right) = S_{\alpha\beta} + 2\sigma n_\alpha p_\beta + n_\alpha n_\beta \kappa$
- where $S_{\alpha\beta} = P_\alpha^{\alpha'}P_\beta^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{\mu\nu}$ $\qquad p_\beta = -n^{\alpha'}P_\beta^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{5\mu}$
  and $\kappa = n^\alpha n^\beta T_{\alpha\beta} \longleftrightarrow T_{55}$ $\qquad T = S + \sigma\kappa.$
- Combining $P_\alpha^{\alpha'}P_\beta^{\beta'}\left(T_{\alpha'\beta'} - \frac{1}{2}g_{\alpha'\beta'}T\right) = S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\left(S + \sigma\kappa\right)$ with $\mathcal{L}_m K_{\alpha\beta}$ leads to the evolution equation for the extrinsic curvature:

$$
\begin{aligned}
\left(\mathcal{L}_5 - \mathcal{L}_{\mathbf{N}}\right)K_{\mu\nu} = &-D_\mu D_\nu N \\
&+N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}
\end{aligned}
$$

Decomposition of the Einstein Equations

Constraint Equations

Projecting Einstein equations twice onto normal n

- Projecting the field equations twice onto the unit normal n and using $n^2 = \sigma$ and the definition of κ leads to $R_{\alpha\beta} n^\alpha n^\beta - \frac{1}{2}\sigma R = \frac{8\pi G}{c^4}\kappa$.

Contracting indices in Gauss relation

- As a relation on $\sum_\tau$, the Gauss relation is:
  $P^{\alpha'}{}_{\alpha} P^{\beta'}{}_{\beta} P^{\gamma}{}_{\gamma'} P^{\delta'}{}_{\delta} R^{\gamma'}{}_{\delta'\alpha'\beta'} = \bar{R}^{\gamma}_{\delta\alpha\beta} - \sigma\left(K^{\gamma}_{\alpha} K_{\beta\delta} - K^{\gamma}_{\beta} K_{\alpha\delta}\right)$
- Expanding the projection operators and contracting indices leads to:
  $R - 2\sigma R_{\alpha\beta} n^\alpha n^\beta = \bar{R} - \sigma\left(K^2 - K^{\alpha\delta} K_{\alpha\delta}\right)$

Hamiltonian Constraint

- Combining the above expressions leads to: $\bar{R} - \sigma\left(K^2 - K^{\mu\nu} K_{\mu\nu}\right) = -\sigma \frac{16\pi G}{c^4}\kappa$

Momentum Constraint

- Projecting the Einstein equations onto $\sum_\tau$ and the normal n and combining with the Codazzi relation $R^5_{\mu\nu\lambda} = \sigma \frac{1}{N}\left(D_\lambda K_{\nu\mu} - D_\nu K_{\lambda\mu}\right)$ leads to:
  $D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4} p_\nu.$

Decomposition of the Einstein Equations

Summary of Differential Equations in 4+1 Formalism

Evolution equation for spacetime metric

$$\frac{1}{c_5}\mathcal{L}_\tau \gamma_{\mu\nu} - \mathcal{L}_{\mathbf{N}} \gamma_{\mu\nu} = -2N K_{\mu\nu}$$

Evolution equation for extrinsic curvature

$$\left(\frac{1}{c_5}\mathcal{L}_\tau - \mathcal{L}_{\mathbf{N}}\right) K_{\mu\nu} = -D_\mu D_\nu N$$
$$+N\left\{-\sigma \bar{R}_{\mu\nu} + K K_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma \frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}$$

Hamiltonian Constraint

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu} K_{\mu\nu}\right) = -\sigma \frac{16\pi G}{c^4}\kappa$$

Momentum Constraint

$$D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Evolution Versus Constraints

Bianchi relation

- In D dimensions the symmetric tensor $G^{\alpha\beta}$ has $D(D+1)/2$ components
- $G^{\alpha\beta}$ has 15 components on $\mathcal{M}_5$
- The Bianchi relation is $\nabla_\alpha G^{\alpha\beta} = 0$ and has 5 components that reduce the independent components of $G^{\alpha\beta}$ to 10.

Order of τ derivative

- The Einstein equations $G_{\alpha\beta} = \dfrac{8\pi G}{c^4} T_{\alpha\beta}$ are differential equations of 2nd order in τ derivatives of $g_{\alpha\beta}$.
- The initial conditions required for evolution of these equations are $\{g_{\alpha\beta}, \partial_\tau g_{\alpha\beta}, T_{\alpha\beta}\}$
- We can rewrite the Bianchi relation in the form:
$$\nabla_\alpha G^{\alpha\beta} = 0 \longrightarrow \frac{1}{c_5}\partial_\tau G^{5\beta} = -\nabla_\mu G^{\mu\beta} + \{\text{Christoffel symbols}\} \times G^{\alpha\beta}$$
- Since the RHS can be at most 2nd order in ∂_τ we see that $G^{5\beta}$ is at most 1st order in ∂_τ.

$G^{5\beta} \longrightarrow$ 5 constraints on initial conditions

- The 5 components of the Bianchi relation are constraints on the initial conditions $\{g_{\alpha\beta}, \partial_\tau g_{\alpha\beta}, T_{\alpha\beta}\}$ and not 2nd order evolution equations.
- Since $G^{\alpha\beta}\big|_{\tau_0} = 0$ it follows that $\dfrac{1}{c_5}\partial_\tau\, G^{5\beta}\big|_{\tau_0} = 0$ and so the constraint is conserved over τ.

Static Schwarzschild-like Geometry

Metric for $T_{\alpha\beta} = 0 \longrightarrow S_{\mu\nu} = p_\mu = \kappa = 0$

- As a first example we consider a static Schwarzschild-like geometry with an additional metric component $g_{55} = \sigma W(x,\tau)$.
- Using the notation of Weinberg we write the squared interval as:
$$ds^2 = -c^2 B dt^2 + A dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \sigma W c_5^2 d\tau^2$$
- where $B(r) = A^{-1}(r) = 1 - \Phi_0(r) = 1 - \dfrac{GM}{rc^2}$,
- the metric takes the form
$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} = \begin{bmatrix} \gamma_{\mu\nu}(r) & 0 \\ 0 & \sigma N^2(x,\tau) \end{bmatrix}$$
- $\bar{R} = 0$ for the Schwarzschild spacetime metric,
- $N^\nu = 0$ and $N = \sqrt{W}$

Dynamical Equations $\longrightarrow$ 4D wave equation for $\sqrt{W}$

- The evolution equation for the spacetime metric is $(\partial_5 - \mathcal{L}_{\mathbf{N}})\,\gamma_{\mu\nu} = -2NK_{\mu\nu}$
- Since $N^\mu = 0$ and $\gamma_{\mu\nu}$ is τ-independent, this becomes $K_{\mu\nu} = 0$.
- The evolution equation for the vanishing extrinsic curvature $K_{\mu\nu}$ is now:
$$\partial_5 K_{\mu\nu} = -D_\mu D_\nu N + N\left(-\sigma \bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda}\right)$$
which is satisfied if W satisfies the wave equation $\gamma^{\mu\nu} D_\mu D_\nu \sqrt{W(x,\tau)} = 0$.

Constraints trivially satisfied for $\bar{R} = K_{\mu\nu} = p_\nu = \kappa = 0$

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa \qquad\qquad D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Perturbation Around Schwarzschild Geometry

Admit variable mass

- As a second example we consider a perturbation around the static Schwarzschild geometry induced by a small variation in the mass M inducing the metric.
- We write $M(\tau) = M[1+\alpha(\tau)]$
  where the perturbation is small so $\alpha^2 \ll 1 \longrightarrow B = A^{-1} = 1 - \Phi_0 [1+\alpha(\tau)]$
- We put $W = 1$.
- The 4D connection is now τ-dependent but retains the unperturbed form so that the space remains flat with $\bar{R} = 0$.

Dynamical equations (neglecting terms in α^2 and Φ_0^2)

- The metric evolution equation is $\partial_5 \gamma_{\mu\nu} = -2NK_{\mu\nu}$
- From the τ derivative of $\gamma_{\mu\nu}$ we solve for
  $$K_{\mu\nu} = -\frac{1}{2c_5}\partial_\tau \gamma_{\mu\nu} = -\frac{\Phi_0 \dot{\alpha}(\tau)}{2c_5}\text{diag}\left(1, \frac{1}{B^2}, 0, 0\right).$$
- Thus we can calculate $K^\mu_\nu = \gamma^{\mu\lambda}K_{\lambda\nu} = -\dfrac{\Phi_0\dot{\alpha}(\tau)}{2c_5}\text{diag}(-1,1,0,0)$ and
  $K = K^\mu_\mu = 0$.
- Using $\bar{R} = 0$, $N = 1$, $N^\mu = 0$, and $(K_{\mu\nu})^2 \propto \alpha^2 \approx 0$, the curvature evolution equation is now
  $$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2c_5^2}\Phi_0\ddot{\alpha}(\tau)\,\text{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}(S + \sigma\kappa)\right]$$
- This equation can be solved for $\alpha(\tau)$ if the energy-momentum tensor is known.

Perturbing Energy Momentum Tensor

To get a feel for the formalism we take $\alpha(\tau)$ to be given and examine the source $S_{\mu\nu}$ that would induce it.

Hamiltonian constraint $\longrightarrow$ no perturbing mass density

- Because $\bar{R} = K = 0$ and $K^{\mu\nu}K_{\mu\nu} \propto \alpha^2\Phi_0^2 \approx 0$ we may take the Hamiltonian constraint as the statement that the mass density $\kappa \approx 0$.

Momentum constraint $\longrightarrow$ perturbing mass current into r direction

- The momentum constraint takes the form
  $p_\nu = D_\mu K^\mu_\nu - D_\nu K = \partial_\mu K^\mu_\nu + K^\lambda_\nu \Gamma^\mu_{\lambda\mu} - K^\mu_\lambda \Gamma^\lambda_{\nu\mu}$ and has components

$$
\begin{aligned}
p_0 &= \partial_r K^1_0 + K^0_0 \Gamma^\mu_{0\mu} - K^\mu_\lambda \Gamma^\lambda_{0\mu} = K^0_0 \Gamma^\mu_{0\mu} - K^0_0 \Gamma^0_{00} - K^1_1 \Gamma^1_{01} = 0 \\
p_1 &= \partial_r K^1_1 + K^\lambda_1 \Gamma^\mu_{\lambda\mu} - K^\mu_\lambda \Gamma^\lambda_{1\mu} = -\frac{1}{2}\frac{1}{c_5 r}\Phi_0 \dot{\alpha}(\tau) \\
p_2 &= p_3 = 0
\end{aligned}
$$

- This corresponds to a mass current flowing into the r direction, driving the varying mass $M(\tau)$.

Evolution equation $\longrightarrow$ perturbing energy density and momentum density in r direction

- The evolution equation is $\ddot{\alpha}(\tau)\ \Phi_0 \mathrm{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = -\sigma \frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} S\right]$ with solution $S_{00} = S_{11} = \left(-\sigma \frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\right)^{-1} \Phi_0 \ddot{\alpha}(\tau) \qquad S_{22} = S_{33} = S = 0$
- This corresponds to a τ-dependent energy density and a energy-momentum flowing into the r direction.