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An Elementary Canonical Classical and Quantum Dynamics, Fourier Transform, Quantum Field Theory and Spin for General Relativity

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Abstract

The basic formulation of the Stueckelberg-Horwitz-Piron classical and quantum theory is described in the framework of special relativity. The theory is then embedded into the manifold of general relativity. The canonical momentum operator is constructed on the manifold, and is seen to have a similar structure to the Newton Wigner position operator in Klein Gordon theory. The Fourier transform on the manifold is then constructed and proved consistent. The quantum theory is then developed in the convenient bra-ket formalism of Dirac. Quantum field theory is then formulated using the construction of a Fock space. Finally, the spin of a particle is defined on the manifold using a theorem of Abraham, Marsden and Ratiu, and its consequences for entanglement are discussed.

0. Introduction

Stueckelberg(1941) wrote down a covariant canonical formulation of classical and quantum mechanics for special relativity with a Hamiltonian on the phase space $\{x^\mu, p_\nu\}$. He envisaged the motion of an event in spacetime generated by dynamical laws as tracing out the worldline of a *particle*. This concept opens the possibility that free motion can carry an event in a straight line in the direction of positive time, but interaction may result in a turning of the line, through a spacelike interval, after which it may proceed in the negative direction of time. Thus the trajectory of the event would not be a single valued function of the time t. He therefore introduced an invariant parameter τ running monotonically along the worldline which then becomes the parameter entering the differential equations of motion. The particle world line corresponding to the motion of the event in

* Part of this work has been reported in (Horwitz 2020)

the negative direction of time is interpreted, following Dirac (as discussed in (Dirac 1947)) as an antiparticle moving in the positive direction of time. The resulting configuration, as stated by Stueckelberg (1941), corresponds to pair annihilation in classical mechanics.

For a Hamiltonian of the form (with metric $(-1, +1, +1, +1)$)

$$K = \frac{p_\mu p^\mu}{2M} + V(x), \tag{0.1}$$

with $V(x)$ a scalar function, the (covariant) Hamilton equations yield

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = \frac{p^\mu}{M} \tag{0.2}$$

and

$$\dot{p}_\mu = -\frac{\partial K}{\partial x^\mu} = -\frac{\partial V(x)}{\partial x^\mu} \tag{0.3}$$

These equations reduce to the usual nonrelativistic form in the nonrelativistic limit. We see from (0.2) that

$$\dot{x}^\mu \dot{x}_\mu = -\frac{ds^2}{d\tau^2} = \frac{p_\mu p^\mu}{M^2}, \tag{0.4}$$

so that the proper time squared ds^2 is equal to $d\tau^2$ when the mass squared $m^2 \equiv= -p_\mu p^\mu$ of the particle is equal to M^2 (called "on shell"). Similar arguments follow when p_μ is replaced by the gauge invariant $p_\mu - eA_\mu$ in the usual electromagnetic form.

Horwitz and Piron (1973) generalized this theory to be applicable to the N body problem (to be called SHP), permitting treatment of many interesting applications (Horwitz and Arshansky 2018). The invariant parameter τ is therefore called the world time.

In this paper, I describe the embedding of this theory into the manifold of general relativity (Horwitz 2019); the proof of the existence of the Fourier transform makes possible the formulation of a quantum theory and quantum field theory on the manifold (Horwitz 2020). We discuss here also a formulation of spin on the manifold and phenomena associated with entanglement.

1. Single particle in external potential

The Hamiltonian of Stueckelbeg, Horwitz and Piron (SHP) (Horwitz 2015a) is

$$K = \frac{1}{2M}\eta^{\mu\nu}\pi_\mu\pi_\nu + V(\xi) \tag{1.1}$$

where $\eta^{\mu\nu}$ is the flat Minkowski metric $(- + ++)$ and π_μ, ξ^μ are the spacetime canonical momenta and coordinates in the local tangent space of the manifold of general relativity (GR).

The existence of a potential term (which may be a Lorentz scalar), representing nongravitational forces, implies that the "free fall" condition is replaced by a local dynamics carried along by the free falling system.

The Hamilton equations are

$$\dot{\xi}^\mu = \frac{\partial K}{\partial \pi_\mu} \qquad \dot{\pi}_\mu = -\frac{\partial K}{\partial \xi^\mu} = -\frac{\partial V}{\partial \xi^\mu}, \tag{1.2}$$

where the dot indicates $\frac{d}{d\tau}$. Since then

$$\begin{aligned} \dot{\xi}^\mu &= \frac{1}{M}\eta^{\mu\nu}\pi_\nu, \\ \text{or} \quad \pi_\nu &= \eta_{\nu\mu} M \dot{\xi}^\mu, \end{aligned} \tag{1.3}$$

the Hamiltonian can be written as

$$K = \frac{M}{2}\eta_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu + V(\xi). \tag{1.4}$$

To carry out the embedding, for $\{x\}$ in the manifold of GR,

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} dx^\lambda \tag{1.5}$$

so that

$$\dot{\xi}^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda}\dot{x}^\lambda. \tag{1.6}$$

The Hamiltonian then becomes

$$K = \frac{M}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + V(x), \tag{1.7}$$

where $V(x)$ is the potential at the point ξ corresponding to the point x and

$$g_{\mu\nu} = \eta_{\lambda\sigma}\frac{\partial \xi^\lambda}{\partial x^\mu}\frac{\partial \xi^\sigma}{\partial x^\nu} \tag{1.8}$$

The corresponding Lagrangian is then

$$L = \frac{M}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - V(x), \tag{1.9}$$

The canonical momentum (da Silva 2006) is gven by

$$p_\mu = \frac{\partial \xi^\lambda}{\partial x^\mu}\pi_\lambda. \tag{1.10}$$

This definition has transformation properties consistent with those of the momentum defined by the Lagrangian (1.9):

$$p_\mu = \frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}, \tag{1.11}$$

yielding

$$p_\mu = Mg_{\mu\nu}\dot{x}^\nu. \tag{1.12}$$

With the usual definition (Weinberg 19720

$$\Gamma^\sigma{}_{\lambda\gamma} = \frac{\partial x^\sigma}{\partial \xi^\mu}\frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma} \tag{1.13}$$

we obtain, from the Hamilton equations applied to the geometrical Hamiltonian the modified geodesic type equation

$$\ddot{x}^\sigma = -\Gamma^\sigma{}_{\lambda\gamma}\dot{x}^\gamma\dot{x}^\lambda - \frac{1}{M}g^{\sigma\lambda}\frac{\partial V(x)}{\partial x^\lambda}, \tag{1.14}$$

Moreover, from (1.12), $p^\mu = Mg^{\mu\lambda}g_{\lambda\nu}\dot{x}^\nu = M\dot{x}^\mu$. Therefore,

$$\dot{p}^\sigma = M\ddot{x}^\sigma = -M\Gamma^\sigma{}_{\lambda\gamma}\dot{x}^\gamma\dot{x}^\lambda - g^{\sigma\lambda}\frac{\partial V(x)}{\partial x^\lambda}, \tag{1.15}$$

a *force* along the geodesic curve.

The Poisson bracket is

$$\begin{aligned}\frac{dF(\xi,\pi)}{d\tau} &= \frac{\partial F(\xi,\pi)}{\partial \xi^\mu}\dot{\xi}^\mu + \frac{\partial F(\xi,\pi)}{\partial \pi_\nu}\dot{\pi}_\mu \\ &= \frac{\partial F(\xi,\pi)}{\partial \xi^\mu}\frac{\partial K}{\partial \pi_\mu} - \frac{\partial F(\xi,\pi)}{\partial \pi_\mu}\frac{\partial K}{\partial \xi^\nu} \\ &\equiv [F,K]_{PB}(\xi,\pi).\end{aligned} \tag{1.16}$$

If we replace in this formula

$$\begin{aligned}\frac{\partial}{\partial \xi^\mu} &= \frac{\partial x^\lambda}{\partial \xi^\mu}\frac{\partial}{\partial x^\lambda} \\ \frac{\partial}{\partial \pi_\mu} &= \frac{\partial \xi^\mu}{\partial x^\lambda}\frac{\partial}{\partial p_\lambda},\end{aligned} \tag{1.17}$$

we obtain

$$\frac{dF(\xi,\pi)}{d\tau} = \frac{\partial F}{\partial x^\mu}\frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial K}{\partial x^\nu} \equiv [F,K]_{PB}(x,p) \tag{1.18}$$

In this definition of Poisson bracket we have, as for the ξ^μ, π_ν relation,

$$[x^\mu, p_\nu]_{PB}(x,p) = \delta^\mu{}_\nu. \tag{1.19}$$

$$[p_\mu, F(x)]_{PB} = -\frac{\partial F}{\partial x^\mu}, \tag{1.20}$$

so that p_μ acts infinitesimally as the *generator of translation* along the coordinate curves* and

$$[x^\mu, F(p)]_{PB} = \frac{\partial F(p)}{\partial p_\mu}, \tag{1.21}$$

* In a geodesically complete manifold, which we shall assume here, the coordinates may be taken to be geodesic curves.

so that x^μ is the generator of translations in p_μ.

We have therefore established a classical canonical mechanics which, following Dirac (1947), can be the basis for a quantum theory, as we discuss below.

2. The many body system with interaction potential

The many body Hamiltonian of the SHP theory is

$$K = \Sigma_{i=1}^N \frac{1}{2M_i}\eta^{\mu\nu}\pi_{\mu i}\pi_{\nu i} + V(\xi_1, \xi_2, \dots \xi_N), \tag{2.1}$$

Following the same procedure as above,

$$K = \Sigma_{i=1}^N \frac{M_i}{2} g_{\mu\nu}(x_i)\dot{x}_i^\mu \dot{x}_i^\nu + V(x_1, x_2, \dots x_N), \tag{2.2}$$

with corresponding Lagrangian

$$L = \Sigma_{i=1}^N \frac{M_i}{2} g_{\mu\nu}(x_i)\dot{x}_i^\mu \dot{x}_i^\nu - V(x_1, x_2, \dots x_N). \tag{2.3}$$

The second order equations for the orbits in spacetime can be written as

$$\ddot{x}_i^\sigma = -\Gamma^\sigma{}_{\lambda\gamma}(x_i)\dot{x}_i^\gamma \dot{x}_i^\lambda - \frac{1}{M_i} g^{\sigma\lambda}(x_i)\frac{\partial V(x_1, x_2, \dots x_N)}{\partial x^\lambda{}_i}. \tag{2.4}$$

Following the same procedure as before, we obtain the canonical bracket

$$[x_i{}^\mu, p_{j\nu}]_{PB} = \delta_{ij}\delta^\mu{}_\nu \tag{2.5}$$

Then, since $\ddot{p}_i^\mu = M_i \ddot{x}_i^\mu$. multiplying by M_i, we obtain

$$\ddot{p}_i^\mu = -\frac{\partial V(x_1, x_2, \dots x_N, p_1)}{\partial x^\mu{}_i} + M_i g_{\sigma\gamma}(x_i)\Gamma^\sigma{}_{\lambda\mu}(x_i)\dot{x}_i^\gamma \dot{x}_i^\lambda. \tag{2.6}$$

3. Quantum Theory on the Curved Space

The Poisson bracket formulas (1.19) and (1.20) can be considered as a basis for defining a quantum theory with canonical commutation relations

$$[x^\mu, p_\nu] = i\hbar\delta^\mu{}_\nu, \tag{3.1}$$

The Stueckelberg-Schrödinger equation for a wave function $\psi_\tau(x)$ is

$$i\frac{\partial}{\partial\tau}\psi_\tau(x) = K\psi_\tau(x), \tag{3.2}$$

where the operator valued Hamiltonian can be taken to be a Hermitian form, on a Hilbert space defined with scalar product (with invariant measure; we write $g = -det\{g^{\mu\nu}\}$),

$$(\psi, \chi) = \int d^4x \sqrt{g} \psi^*{}_\tau(x) \chi_\tau(x). \tag{3.3}$$

It is easily seen that the operator

$$p_\mu = -i \frac{\partial}{\partial x^\mu} - \frac{i}{2} \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\mu} \sqrt{g(x)} \tag{3.4}$$

is essentially self-adjoint in the scalar product (3.3), satisfying as well the commutation relations (3.1). This operator is similar in form to the Newton-Wigner consruction of the coordinate operator in Klein-Gordon theory, and in the same way can be transformed back to a simple derivative by a Foldy-Wouthuysen type transformation as we shall do below.

Since p_μ defined by (3.4) is Hermitian in the scalar product (3.3), we can write the Hermitian Hamiltonian as

$$K = \frac{1}{2M} p_\mu g^{\mu\nu} p_\nu + V(x). \tag{3.5}$$

This construction can be carried over to the many body case directly, *i.e*, with the operator properties of the coordinates and momenta

$$[x^\mu{}_i, p_{\nu i}] = i\hbar \delta^i{}_j \delta^\mu{}_\nu, \tag{3.6}$$

The scalar product is then

$$(\psi, \chi) = \int \Pi_i \{d^4(x_i) \sqrt{g(x_i)}\} \psi^*{}_\tau(x_1, x_2, \ldots x_N) \chi_\tau(x_1, x_2, \ldots x_N). \tag{3.7}$$

In this scalar product, the Hamiltonian (with (3.4) for each $p_{\mu i}$ at $x^\mu{}_i$)

$$K = \Sigma_i \frac{1}{2M_i} p_{\mu i} g^{\mu\nu}(x_i) p_{\nu i} + V(x_1, x_2, \ldots x_N) \tag{3.8}$$

is essentially self-adjoint.

4. Electromagnetism

To satisfy the reqirement of invariance under local gauge transformations (Horwitz 2015a)

$$(p_\mu - a'_\mu(x,\tau)) e^{i\Lambda(x,\tau)} \psi_\tau(x) = e^{i\Lambda(x,\tau)} (p_\mu - a_\mu(x,\tau)) \psi_\tau(x). \tag{4.1}$$

Unless we restrict ourselves to the so-called "Hamilton gauge" (with Λ independent of τ), the form of the Stueckelberg-Schrödinger implies the existence of a fifth field (Saad 1989) $a_5(x,\tau)$, for which we must have (in close analogy to the generation of A_0 field in the electromagnetism associated with the nonrelativistic Schrödinger equation)

$$a'_5(x,\tau) = a_5(x,\tau) + \frac{\partial}{\partial \tau} \Lambda(x,\tau). \tag{4.2}$$

The Stueckelberg-Schrödinger equation then becomes

$$i\frac{\partial}{\partial\tau}\psi_\tau(x) = \{\frac{1}{2M}(p_\mu - a_\mu(x,\tau))g^{\mu\nu}(p_\nu - a_\nu(x,\tau)) - a_5(x,\tau)(x)\}\psi_\tau(x), \tag{4.3}$$

where the scalar field of the potential model is now replaced by the generally τ dependent $a_5(x,\tau)$. This field plays an important role in the study of the self-interaction problem (Aharonovich 2012).

5. Fourier analysis

To define a consistent quantum theory, we must define a Fourier transform on the manifiold. For a function $f(x)$ defined almost everywhere on the manifold $\{x\}$, we define the Fourier transform

$$\tilde{f}(p) = \int d^4x\sqrt{g}\ e^{-ip_\mu x^\mu} f(x), \tag{5.1}$$

where $g = -\det g_{\mu\nu}$ and the integral is carried out (in the Riemannian sense) in the limit of the sum over small spacetime volumes with invariant measure $d^4x\sqrt{g}$. The inverse is

$$f(x) = \frac{1}{(2\pi)^4}\frac{1}{\sqrt{g(x)}}\int d^4p e^{ip_\mu x^\mu}\tilde{f}(p). \tag{5.2}$$

Note that $p_\mu x^\mu \equiv -p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3$ is not local diffeomorphism invariant, and hence not a scalar product, on the manifold. The Fourier transform as we have defined it is carried out in the framework of a given, arbitrary, coordinatization.

Provided that

$$\int d^4p\ e^{-ip_\mu(x^\mu - x'^\mu)} = (2\pi)^4\delta^4(x-x'), \tag{5.3}$$

so that

$$(2\pi)^{-4}\int d^4x'\int d^4p\ e^{-ip_\mu(x^\mu - x'^\mu)} = 1, \tag{5.4}$$

we must have, for consistency,

$$\tilde{f}(p) = \frac{1}{(2\pi)^4}\int d^4x\int d^4p'\ e^{-i(p_\mu - p'_\mu)x^\mu}\tilde{f}(p'). \tag{5.5}$$

We must therefore study the function (in a particular coordinatization $\{x\}$ and cotangent space $\{p\}$)

$$\Delta(p-p') \equiv \frac{1}{(2\pi)^4}\int d^4x\ e^{-i(p_\mu - p'_\mu)x^\mu} \tag{5.6}$$

which should act as the distribution $\delta^4(p-p')$.

To prove this consistency condition, following the method of Reed and Simon (Reed 1972) in their discussion of Lebesgue integration, we represent the integral as a sum over small boxes around the set of points $\{x_B\}$ that cover the space (which we have assumed to be non-compact), and eventually take the limit as for a Riemann-Lebesgue integral.

In each small box, the coordinatization arises from an invertible transformation from the local tangent space in that neighborhood. We write

$$x^\mu = x_B{}^\mu + \eta^\mu \qquad \in \mathrm{boxB} \tag{5.7}$$

where

$$\eta^\mu = \frac{\partial x^\mu}{\partial \xi^\lambda} \xi^\lambda \tag{5.8}$$

and ξ^λ (small) is in the flat local tangent space at x_B.

We now write the integral (5.6) as

$$\begin{aligned}\Delta(p-p') &= \frac{1}{(2\pi)^4} \Sigma_B \int_B d^4\eta \; e^{-i(p_\mu - p'_\mu)(x_B{}^\mu + \eta^\mu)} \\ &= \frac{1}{(2\pi)^4} \Sigma_B e^{-i(p_\mu - p'_\mu) x_B{}^\mu} \int_B d^4\eta \; e^{-i(p_\mu - p'_\mu)\eta^\mu}.\end{aligned} \tag{5.9}$$

Let us call the measure at B

$$\boldsymbol{\Delta}\mu(B, p-p') = \int_B d^4\eta \; e^{-i(p_\mu - p'_\mu)\eta^\mu}. \tag{5.10}$$

In the neighborhood of each B, define

$$\frac{\partial x^\mu}{\partial \xi^\lambda} = \frac{\partial \eta^\mu}{\partial \xi^\lambda} \equiv a^\mu{}_\lambda(B), \tag{5.11}$$

which may be taken to be a constant matrix in each small box. In (5.11), we then have

$$\boldsymbol{\Delta}\mu(B, p-p') = \det a \int_B d^4\xi \; e^{-i(p_\mu - p'_\mu) a^\mu{}_\lambda(B)\xi^\lambda}. \tag{5.12}$$

We now make a change of variables for which $\xi'^\mu = a^\mu{}_\lambda(B)\xi^\lambda$; then, since $d^4\xi' = \det a d^4\xi$, we have

$$\boldsymbol{\Delta}\mu(B, p-p') = \int_{B'(B)} d^4\xi' \; e^{i(p_\mu - p'_\mu)\xi^{\mu'}}. \tag{5.13}$$

in each box.

We may then write (5.9) as

$$\Delta(p-p') = \frac{1}{(2\pi)^4} \Sigma_B \; e^{-i(p_\mu - p'_\mu) x_B{}^\mu} \boldsymbol{\Delta}\mu(\xi_B, p = p') \tag{5.14}$$

and consider the set $\{x_B\}$ to be in correspondence with an extended flat space $\{\xi(x_B)\}$, to obtain***

$$\Delta(p-p') = \frac{1}{(2\pi)^4} \Sigma_B \; e^{-i(p_\mu - p'_\mu)\xi_B{}^\mu} \boldsymbol{\Delta}\mu(\xi_B, p-p'). \tag{5.15}$$

*** This procedure is somewhat similar to the method followed in the simpler case of constant curvature by Georgiev who, however, used eigenvalues of the Laplace-Beltrami operator.

Note that the transformation to the local tangent space is, in general, different in each neighboring box. There would therefore be a volume deficit in the sum which may be divergent. This problem can be solved by constructing neighboring boxes by parallel transport along and orthogonal to geodesic curves assuming, as we have remarked above, geodesic completeness. In the limit of small spacetime box volume, the sum (5.15) approaches a Lebesgue type integral on a flat space

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\int e^{-i(p_\mu - p'_\mu)\xi^\mu} d\mu(\xi, p-p'). \tag{5.16}$$

If the measure is differentiable, we could write,

$$d\mu(\xi, p-p') = m(\xi, p-p')d^4\xi. \tag{5.17}$$

In the small box, say, size ϵ, one can prove, in the limit, that $m = 1$, so that

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\int e^{-i(p_\mu - p'_\mu)\xi^\mu} d^4\xi, \tag{5.18}$$

or

$$\Delta(p-p') = \delta^4(p-p'). \tag{5.19}$$

6. Consequences for the Quantum Theory

The scalar product for the SHPGR Hilbert space (Horwitz 2019) is

$$(\psi, \chi) = \int d^4x\sqrt{g}\psi^*{}_\tau(x)\chi_\tau(x). \tag{6.1}$$

Replacing all wave functions $\psi(x)$ by $g(x)^{\frac{1}{4}}\psi(x)$, which we shall call the Foldy-Wouthuysen representation in coordinate space, the operator (3.4) becomes just $-i\frac{\partial}{\partial x^\mu}$.

To cast our results in the familiar form of the quantum theory, we write the scalar product (6.1) as

$$< \psi|\chi >= \int d^4x < \psi|x >< x|\chi >, \tag{6.2}$$

where

$$\begin{aligned} < x|\chi > &= g(x)^{\frac{1}{4}}\chi(x) \\ < x|\psi > &= g(x)^{\frac{1}{4}}\psi(x), \end{aligned} \tag{6.3}$$

(and $< \psi|x >=< x|\psi >^*$) consistently with (6.2). We now wish to show that the Parseval-Plancherel (Parseval 1806)(Plancherel 1910)relation holds for the momentum representation for the integral (6.2).

We define

$$< x|p >= \frac{1}{(2\pi)^4 g(x)^{\frac{1}{4}}} e^{ip_\mu x^\mu} \tag{6.4}$$

and

$$< p|x >= g(x)^{\frac{1}{4}} e^{-ip_\mu x^\mu}, \tag{6.5}$$

which also follows from considering the ket $|p>$ as a limiting case of a sharply defined function $\tilde{f}(p)$ in (5.2) (but in Foldy-Wouthuysen representation). With (6.5) we have

$$\int d^4p < x|p >< p|x' >= \delta^4(x - x'). \tag{6.6}$$

It then follows from (6.5) that

$$\begin{aligned} < p|\chi > &= \int d^4x < p|x >< x|\chi > \\ &= \int d^4x \; g(x)^{\frac{1}{4}} \; e^{-ip_\mu x^\mu} g(x)^{\frac{1}{4}} \chi(x) \qquad (6.7). \\ &= \int d^4x \; e^{-ip_\mu x^\mu} \sqrt{g(x)} \chi(x) = \tilde{\chi}(p). \end{aligned}$$

Moreover, from (6.4),

$$\begin{aligned} < \psi|p > &= \int d^4x < \psi|x >< x|p > \\ &= \int d^4x \; g(x)^{\frac{1}{4}} \psi^*(x) \frac{1}{2\pi^4 g(x)^{\frac{1}{4}}} e^{ip_\mu x^\mu} \\ &= \int \frac{d^4x}{(2\pi)^4} \; e^{ip_\mu x^\mu} \psi^*(x) \neq \tilde{\psi}^*(p), \end{aligned} \tag{6.8}$$

with $\tilde{\psi}(p)$ as would be defined in (5.1). Note that this is the complex conjugate of $< p|\psi >$ only in the flat space limit, reflecting the structure of (6.7) and (6.8). This function, however, serves as the *dual* of the function $< p|\psi >$ for the construction of the scalar product contracting, for example, with $< p|\chi >$ to give the scalar product, and for $\chi = \psi$, the norm.

From (6.7) and (6.8), we have

$$\begin{aligned} \int d^4p < \psi|p >< p|\chi > = \int d^4p \int \frac{d^4x}{(2\pi)^4} \\ \times e^{ip_\mu x^\mu} \psi^*(x) \int d^4x' \; e^{-ip_\mu x'^\mu} \sqrt{g(x')} \chi(x') \\ = \int d^4x \sqrt{g(x)} \; \psi^*(x) \chi(x). \end{aligned} \tag{6.9}$$

This, in fact, completes our explicit proof of the Parseval relation

$$\int d^4x \sqrt{g(x)} \; |\psi(x)|^2 = \int d^4p < \psi|p >< p|\psi > . \tag{6.10}$$

Note that $<\psi|p><p|\psi>$ is not necessarily a positive number; only the integral assures positivity and unitarity of the Fourier transform, since $<\psi|p>$ is not the complex conjugate of $<p|\psi>$.

As pointed out above, the operator $p_\mu = -i\frac{\partial}{\partial x^\mu}$ is essentially self-adjoint in the Foldy-Wouthuysen representation. We now examine its spectrum. We use the notation $\{X\}$ and $\{P\}$ to distinguish the canonical operators from the numerical parameters.

Since, by definition, we should have

$$< x|P_\mu|\psi >= -i\frac{\partial}{\partial x^\mu} < x|\psi > \tag{6.11}$$

we have, by completeness of the spectral family of X,

$$P_\mu|\psi >= \int d^4x|x > \left(-i\frac{\partial}{\partial x^\mu}\right) < x|\psi >, \tag{6.12}$$

giving P in operator form in the $x-$representation. In $p-$representation, we have, using the transformation functions above,

$$\begin{aligned}\int d^4x < p|x > P_\lambda < x|p' > &= \int d^4x\ e^{-ip_\mu x^\mu}\left(g(x)^{\frac{1}{4}}P_\lambda g(x)^{-\frac{1}{4}}\right)e^{ip'_\mu x^\mu} \\ &= p_\lambda \delta^4(p-p'),\end{aligned} \tag{6.13}$$

where we recognize the central factor in parentheses as the Foldy-Wouthuysen form of the momentum operator.

Finally,in the same way, for the canonical coordinate, we should have

$$< p|X^\mu|\psi >= i\frac{\partial}{\partial p_\mu} < p|\psi > . \tag{6.14}$$

Then, in the x representation (X^λ commutes with $g(x)$),

$$\begin{aligned}\int d^4p < x|p > X^\lambda < p|x' > &= \int d^4p\frac{1}{(2\pi)^4}e^{ip_\mu x^\mu}X^\lambda e^{-ip_\mu x'^\mu} \\ &= x^\lambda\delta^4(x-x').\end{aligned} \tag{6.15}$$

7. Quantum Field Theory

To define a quantum field theory on the curved space, we shall construct a Fock space for the many body theory in terms of the direct product of single particle states in momentum space (Horwitz and Arshansky 2018), and define creation and annihilation operators using the definition of the scalar product. The Fourier transform of these operators is then used to construct the quantum fields. We have seen in the previous section that for the state $\psi(x)$ of a one particle system, the complex conjugate of the state (we suppress the tilde in the following) in momentum space

$$\psi^*(p) = \int d^4x \ e^{ip_\mu x^\mu} \sqrt{g(x)}\psi^*(x) \tag{7.1}$$

is not equal to the dual $< \psi|p >$

$$\psi^\dagger(p) = \frac{1}{(2\pi)^4} \int d^4x \ e^{ip_\mu x^\mu} \psi^*(x). \tag{7.2}$$

Here, the dagger is used to indicate the vector *dual* to $\psi(x)$, necessary, as in Eq. (6.5), to form the scalar product. In this form, (6.9) can be written as

$$\int d^4p\psi_1{}^\dagger(p)\psi_2(p) = \int d^4x\sqrt{g(x)} \ \psi_1{}^*(x)\psi_2(x). \tag{7.3}$$

In this sense, $< p|x >$ (of (6.5)) is the dual of the (generalized) momentum state $< x|p >$ in the Foldy-Wouthuysen representation. The operator representations (6.13) and (6.15) are therefore bilinears in the states and their duals, and, as shown below, correspond to the second quantized form of the operators, as in the usual form of "second quantization". Note that the linear functional $L(\psi)$ of the Riesz theorem (Riesz 1955) that reaches a maximum for a given ψ_0 is given uniquely by the scalar product (7.3), $L(\psi) = \int d^4p\psi^\dagger(p)_0\psi(p)$.

The many-body Fock space is constructed by representing the N-body wave function for identical particles on the basis of states of the form, here suitably symmetrized for Bose-Einstein or Fermi-Dirac statistics at equal τ,* In the following we work out the Fermi-Dirac case explicitly;the Bose-Einstein formulation is similar. We define, for the Fermi-Dirac case,

$$\Psi_{N,\tau}(p_N, p_{N-1}, \ldots p_1) = \frac{1}{N!}\Sigma(-1^P P \ \psi_N \otimes \psi_{N-1} \otimes \ldots \otimes \psi_1)(p_N, p_{N-1} \ldots p_1), \tag{7.4}$$

where all states in the direct product are at equal τ (with *e.g.* $\Psi_2 = \frac{1}{2}(\psi_2 \otimes \psi_1 - \psi_1 \otimes \psi_2)(p_2, p_1) = \frac{1}{2}(\psi_2(p_2)\psi_1(p_1) - \psi_1(p_2)\psi_2(p_1)))$. We work initially in momentum space, since in this representation the structure is most similar in form to the usual construction.

The Fock space consists of span of the set of the form (7.4), for every $N = (0, 1, \ldots \infty)$, where $N = 0$ is the vacuum state. We now define the creation operator $a^\dagger(\psi_{N+1}$ on this space with the property that**

$$\Psi_{N+1}(p_{N+1}.p_N, \ldots p_1) = a^\dagger(\psi_{N+1})\Psi_N)(p_N, p_{N+1}, \ldots p_1), \tag{7.5}$$

* Other approaches to quantum field theory on the manifold of general relativity, such as in (Birrell 1982), introduce a timelike foliation of space time to describe the fields and their evolution. There is no necessity for us to do this since we have available the universal invariant parameter τ. The usual specification of a spacelike surface on which a complete set of local observables $\{\mathcal{O}_\tau(x)\}$ commute is then correlated to this τ.

** Here the dagger indicates the Hermitian conjugate in the Fock space scalar product. We use this notation because its Fock space Hermitian conjugate carries the dual vector to the scalar product.

which carries out as well the appropriate antisymmetrization. In order to define the annihilation operator we take the scalar product of this state with an $N+1$ particle state

$$\Phi_{N+1,\tau}(p_{N+1},p_N,\dots p_1)=\frac{1}{(N+1)!}\Sigma(-1^P P\phi_{N+1}\otimes\phi_N\otimes\dots\otimes\psi_1)(p_{N+1},p_N\dots p_1), \quad (7.6)$$

for which

$$(\Phi_{N+1},a^\dagger(\psi_{N+1})\Psi_N)=(a(\psi_{N+1})\Phi_{N+1},\Psi_N) \quad (7.7)$$

where $a(\psi_{N+1})$, the Hermtian conjugate of $a^\dagger(\psi_{N+1})$ in the Fock space, is an annihilation operator that removes the particle in the state ψ_{N+1}. This scalar product is defined on the momentum space by (7.3) term by term, using the dual vectors $\psi^\dagger$, as in (7.3), thus defining the adjoint on the Fock space.

8. Spin of a Particle in SHPGR and Entanglement

The theory of intrinsic angular momentum of a particle in the framework of relativistic quantum theory for special relativity was worked out by Arshansky and Horwitz (Arshansky 1989c)following the method of induced representations of Wigner (Wigner 1939) but with an inducing timelike vector n^μ, transforming with the Lorentz group, independent of momentum. The reason for using n^μ instead of p^μ is that the induced representation of the angular momentum on the wave function depends on the inducing vector. When computing the expectation value of x^μ, represented (in momentum space) by $i\frac{\partial}{\partial p_\mu}$ in the relativistic quantum theory, this derivative would destroy the unitarity of a representation induced on p^μ. This expectation value would then not transform as a vector under the Lorentz group.

The generators of the Lorentz group acting both on $\{x^\mu\}$ and $\{n^\mu\}$ are, in the special relativistic quantum theory

$$\begin{aligned} M^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\nu \\ &\quad - i\big(n^\mu\frac{\partial}{\partial n_\nu}-n^\nu\frac{\partial}{\partial n_\mu}\big), \end{aligned} \quad (8.1)$$

where indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}=(-1,+1,+1,+1)$. Under the action of the group generated by this set of operators, $M^{\mu\nu}$ is a Lorentz tensor. It is essential for the embedding of the special relativistic theory into GR that the set of local generators transform under the local embedding diffeomorphisms as covariant tensors. This is clearly true for the $\{x,p\}$ part of $M^{\mu\nu}$(for support of the wave function in GR on small x^μ); we shall see by the isomorphism theorem of Abraham, Marsden and Ratui (Abraham 1988), that n^μ transforms contravariantly as well, preserving its timelike character, under local diffeomorphisms as well.

The spin of a particle is an essentially quantum mechanical property. In the non-relativistic quantum theory, the lowest non-trivial representation of the rotation group corresponds to the spin degrees of freedom of the particle. However, for a relativistic particle, the Lorentz group $O(3,1)$ or its covering $SL(2,C)$ acts on the wave function.Wigner

[17] showed that such a representation can be constructed by starting with a particle at rest so that its four-momentum has just one component, $p_0 = m$, where m is the mass of the particle (assumed nonzero; the zero mass case, such as for the photon,must be treated separately). In the four dimensional Minkowsi space this vector lies along the time axis. The elements of the Lorentz group that leave this vector invariant lie in the subgroup $SO(3)$ or its covering $SU(2)$, and therefore provide a representation of spin in that frame. Under a Lorentz boost the vector $(p_0, 0, 0, 0)$ may move to a general timelike four vector p_μ, but the action of the group remains the same about this new vector, *i.e.*, it remains in $SU(2)$. This so-called *induced representation* is then identified with the intrinsic spin of the particle. The structure of higher than spin 1/2 states was worked out by Horwitz and Zellig-Hess (Horwitz 2015b) by means of tensor products in the representation space. As pointed out there, this mathematical construction can be considered as well to apply to the total spin states of a many-body system, and, in particular to the spin zero and one states of a two body system, each with spin 1/2. This coupling of spins is independent of the spacetime coordinates of the two particles, and therefore provides a basis for entanglement that can be carried, as we argue below, to the context of general relativity.

We now wish to imbed this structure into the manifold of GR. We first remark that since the commutation relations of the canonical coordinates and momenta remain the same under the embedding, the first term in (8.1) remains a valid generator of the Lorentz group. Its meaning for the particle as an intrinsic property requires that the wave function has sufficiently small support in spacetime.

We now use a theorem stated in Abraham, Marsden and Ratiu (Abraham 1988):

Under the C^r map φ, for X, Y elements of an algebra on an r-manifold, $X \to X'$ and $Y \to Y'$, f a function on the manifold,

$$([X', Y'][f]) \circ \varphi = [X, Y][f \circ \varphi \tag{8.2}$$

establishing an algebraic isomorphism.

In our case,

$$\varphi : \psi_n(\xi) \to \psi'_{n'}(x) \tag{8.3}$$

Defining the angular momentum in a small neighborhood so that the variables ξ can be considered to be very small, and defining

$$n'\mu = \frac{\partial x^\mu}{\partial \xi^\lambda} n^\lambda, \tag{8.4}$$

under the local diffeomorphism

$$\varphi : [M_\xi{}^{\mu\nu}, M_\xi{}^{\alpha\beta}] \to [M_x{}^{\mu\nu}, M_x{}^{\alpha\beta}]. \tag{8.5}$$

The Lorentz algebra therefore remains under these local diffeomorphisms, and we can follow the construction of the induced representation for spin just as in the flat Minkowski space. Moreover, as for the tensor product methods for the construction of higher spin states, we may consider the representations as corresponding, in the spin space, to the tensor products of the states of a many body system, without reference to the coordinatization of the wave functions. This establishes a basis for entanglement in general relativity.

In particular, we have shown that the momentum operator generates translation along the coordinates. Choosing coordinates along geodesic curves, we can achieve in this way parallel transport. For two elementary systems, for example, with spin $1/2$, one can consruct in this way, as argued above. a singlet state of the two body system with the properties of the Einstein-Podolsky-Rosen (EPR)(Einstein 1935) construction. In this way, two particles initially in a spin zero state in the spin space, with wave packets moving coherently along geodesic curves, should maintain the EPR correlations.

An interesting associated question is that of transport of a single particle along a closed geodesic curve when a gravitational field is enclosed, inducing a change of orientation of the spin, leading to a Bohm-Aharonov (Aharonov 1959) type effect of interference in the spin space.

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