Study of the Radiation Reaction Force for a Step Electric Field and an Electromagnetic Pulse

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Abstract. The motions of a spin-less point-like charged particle predicted by the Landau-Lifshitz equation and the Hammond method are obtained for a step electric field, a smooth step electric field and an electromagnetic pulse by using analytical and numerical solutions. In addition to Hammond method not presenting the so-called constant force paradox, using step force brings out the apparent physical contradictions of Landau-Lifshitz equation regarding energy conservation. Nevertheless, a smooth step force shows the consistency of the Landau-Lifshitz equation. Unlike other cases, the electromagnetic pulse shows another fundamental difference between the two models. Finally, an analysis of the Hammond method is made.

1. Introduction

In recent years, the Landau-Lifshitz equation [1] [LL] has been considered by many authors as the best equation, among many others [2], [3], [4] (see also Parrot book [5] and references therein), to describe the motion of a spin-less point-like charged particle including the radiation reaction force within the framework of Classical Electrodynamics. Although for some authors [5], the expression for the LL radiation reaction force is an approximation or a variant of Lorentz-Dirac equation [6] [LD], for others, it represents an exact one [7], [8], [9], [10], [11], [12], [13]. Being a second-order differential equation, it does not present solutions with physical anomalies such as self-accelerations and pre-accelerations that exist in Dirac's theory. Therefore, using the LL equation it is not necessary to use the asymptotic conditions required in Dirac's theory. It should be noted that in recent years ultrahigh intensity electromagnetic pulses ($10^{22}\ Wcm^{-2}$ [14]) have been produced making the analysis of the radiation reaction term more important. It has been shown that for such pulses, different equations as the Eliezer-Ford-O'Connell [15], [16] [EFO] and LL equations predict the same effects [17], [18]. Moreover, using the LL equation, it is possible to carry out a deeper study of a relativistic plasma modifying the relativistic Vlasov equation and obtaining new dispersion relations other than those deduced without considering radiation reaction term [19]. On the other hand, DeWitt and Brehme [20] obtained a generalization of the LD equation in General Relativity which was corrected by Hobbs [21] few years later. Moreover, the corresponding generalization in General Relativity of the LL equation was made by Quinn and Wald [22]. Also, considering a charged particle with structure, Ford and O'Connell [23], [16], [24] by using quantum arguments and a Langevin equation, deduced an equation of motion for the non-relativistic case, known as the Ford equation which can be physically generalized to Special Relativity giving the Eliezer equation ; that is the EFO equation [15], [16]. Moreover, Krivitskiĭ and Tsytovich [25] shown that the LL radiation reaction term represents an average radiation reaction force in Quantum Electrodynamics. This last point reinforces the idea that the LL equation can be taken as the equation that describes the motion of a spin-less point-like charged particle in Classical Electrodynamics.

Although the LL equation possesses all the physical qualifications to describe the motion of a charged particle, when the solution is analyzed in the case of a constant electric field it turns out that the radiation reaction force vanishes [8], [26], [27]. That is: for a constant force, there is no effect in the motion of the charged particle due to the radiation reaction force and the charged particle is driven by the Lorentz force. However, DeWitt and Brehme [20] explained this phenomenon by noticing that the radiation exits at the infinite; that is, the energy radiated to infinite is taken from the attached fields (The Scott term or the acceleration energy) and consequently even if the total radiation reaction term in the equation of motion vanishes, the radiation to infinite (the irreversible emission of radiation) exists. This has been calculated and proven by Ares de Parga [12]. However, this explanation is not accepted by many authors [28], [29] and they have reached the conclusion that the rest mass of the charge is not conserved. Moreover, Sorkin [30] says that this phenomenon raises a paradox which we will call the constant electric field paradox. Recently, Hammond [31], [32], [33], [34], [35], [36], [37], proposed a new method which avoids this paradox. This method consists of searching an expression for the radiation reaction force for each applied force to the charge and it is solved in many cases by using an iteration model of equations which gives a result at first order in $\tau_o = 2q^2/3mc^3$ (the characteristic time of the charge). The results practically coincides with the solutions of the LL and EFO equations in many cases. Indeed, if we make a comparison of the solutions of both, the LL equation and the Hammond method, for the constant magnetic field [38], [32], [33], for the central field [39] and for the low energy electromagnetic pulses, we notice that within the approximations made for the levels of energy where the damping force is important, the results are similar. However, Hammond [35] claims that for ultrahigh intensity electromagnetic pulses the differences between the motions predicted by the EFO equation, the LL equation and the H method are important. Such differences appear within the Shen's zone [40] where quantum effects are not important and an equation of motion is meaningful for a physical description. Moreover, Hammond argues that the results obtained by using LL equation do not accomplish a balance of energy [35]. In counterpart, the LL and the EFO equations are founded in different expressions for the radiated energy at the infinite; that is: the Larmor formula does not represent the radiation power at the infinite in these theories [12], [13].

It has to be noticed that the Lawson-Woodward theorem [41], [42] states, if radiation reaction is excluded, the particle gains no net momentum from the pulse. This happens when we consider the Lorentz solution. We will expect that when the radiation reaction force is included in any of the different equations that we are dealing with, a gain of energy must appear, particularly in the direction of the pulse. This represents the reason to study the effects of the different reaction terms when an ultrahigh intensity electromagnetic pulse is applied.

Although in many cases the solutions of Hammond method and those of LL equation coincide, in the cases of the constant electric field (including the step and the smooth step electric field) and the ultrahigh intensity electromagnetic pulse, there are differences. Indeed, we will see in this article that solving both methods in the two mentioned cases such differences appear. However, such differences differ from those described by Hammond. The purpose of this article consists of showing that the Landau-Lifshitz equation represents the better equation to describe the motion of a spin-less point-like charged particle in Classical Electrodynamics.

The article is organized as follows. In section 2, By using the LL equation, the constant electric field paradox is described. The Hammond method is exposed and the result obtained by applying it to the constant electric field is compared with the ones obtained for the same case from the L and LL equations. In section 3, the results obtained for the step electric field case by using the L and the LL equations, and the H method, are compared. Since, a step electric field is not physically acceptable, the same calculations and comparisons are made for the smooth step electric field. In section 4, the electromagnetic pulse case is analyzed for the L equation, for the LL equation, for a proposed Lorentz-Dirac-Hammond equation [LDH] and the Hammond approximation equation [HA]. The technique used by Hammond for obtaining the numerical solution of the LL equation in the case of the ultrahigh intensity electromagnetic pulse is also analyzed and compared with our result. In section 5, a discussion is made about the pros and cons of the two theories showing the inherent defect of Hammond's method. Some concluding remarks are made in section 6.

2. The Constant Electric Field Paradox and the Hammond Method

The constant electric paradox consists of having a vanishing radiation reaction term in the equation of motion of a spin-less point-like charged particle when a constant electric field or constant force is applied to the charge. These happens when we deal with the LD, the EFO or the LL equations.

2.1. The Landau-Lifshitz Equation and the Constant Electric Field Paradox

Since we are interested in comparing the LL equation with the H method, let us begin by expressing the LL equation [1]:

$$\begin{aligned} ma^{\mu} &= (q/c)F^{\mu\nu}w_{\nu} \\ &+\tau_o\left[\frac{q}{c}(\frac{\partial F^{\mu\nu}}{\partial x^{\alpha}}w^{\alpha}w_{\nu}-(q/cm)F^{\mu\nu}F_{\alpha w}w^{\alpha})+(q^2/c^4m)F^2w^{\mu}\right]. \end{aligned} \tag{1}$$

By defining [19],

$$\Delta^{\mu\nu}(w)=g^{\mu\nu}-\frac{w^{\mu}w^{\nu}}{c^2}, \tag{2}$$

we obtain

$$ma^{\mu}=\frac{e}{c}F^{\mu\nu}w_{\nu}+m\tau_o\Delta^{\mu\nu}(w)\frac{e}{mc}\left[\frac{e}{mc}F_{\nu\alpha}F^{\alpha\beta}w_{\beta}+w^{\rho}w^{\alpha}\frac{\partial F_{\nu\alpha}}{\partial x^{\rho}}\right]. \tag{3}$$

With this expression of the LL equation, it is easy to show that the radiation reaction term vanishes when we applied a constant electric field. Consider the radiation reaction term for a constant electric field E,

$$\begin{aligned} &\Delta^{\mu\nu}(w_{\rho})\left[\frac{e}{mc}\left[F_{\nu\alpha}\right]\left[F^{\alpha\beta}\right]w_{\beta}+w^{\rho}w^{\alpha}\frac{\partial F_{\nu\alpha}}{\partial x^{\alpha}}\right] \\ =\;&\Delta^{\mu\nu}(w_{\rho})\left[\frac{e}{mc}\left[F^{ext}_{\nu\alpha}\right]\left[F^{\alpha\beta}\right]w_{\beta}\right]. \end{aligned} \tag{4}$$

Then,

$$\Delta^{\mu\nu}(w_{\rho})\left[\frac{e}{mc}\left[F_{\nu\alpha}\right]\left[F^{\alpha\beta}\right]w_{\beta}+w^{\rho}w^{\alpha}\frac{\partial F_{\nu\alpha}}{\partial x^{\alpha}}\right]$$

$$
\begin{aligned}
&= (n^{\mu\nu} - \frac{w^\mu w^\nu}{c^2}) \times \left[\frac{e}{mc} \left[F_{\nu\alpha} \right] \left[F^{\alpha\beta} \right] w_\beta \right] \\
&= E^2 w^\mu \left(1 - \frac{c^2}{c^2} \right) = 0. \qquad (5)
\end{aligned}
$$

Therefore, for the constant electric force, the LL equation is equivalent to the L equation of motion and there is no effect on the motion of the charge due to the radiation reaction force. Notice that this does not mean that the charge does not radiate. It means that the radiated energy to the infinity is taken from the attached fields [12], [26], [20].

In order to make the comparison with the H method, let us deduce the solution of this case. If we consider the electric field in the x^1 direction, the L and LL equations turn to be

$$
\begin{aligned}
\frac{dw^0}{d\tau} &= \frac{eE}{mc} w^1 = \Omega w^1 \\
\frac{dw^1}{d\tau} &= \frac{eE}{mc} w^0 = \Omega w^0, \qquad (6)
\end{aligned}
$$

where $\Omega = \frac{eE}{mc}$. If we impose the initial conditions for the 4−velocity, $w^0 = c$ and $w^1 = 0$, the well-known solutions are:

$$
\begin{aligned}
w^0 &= c \cosh \Omega\tau \\
w^1 &= c \sinh \Omega\tau. \qquad (7)
\end{aligned}
$$

These solutions are described in figures (1) and (2).

2.2. The Hammond Method

The constant force paradox encouraged Hammond to develop a theory which avoids it [31], [32], [33], [34], [35], [36]. He began by proposing an equation of this type

$$
\frac{dw^\mu}{d\tau} = \frac{e}{mc} F^{\mu\sigma} w_\sigma + f^\mu, \qquad (8)
$$

where the radiation reaction force f^μ is described by

$$
f^\mu = \phi^{,\mu} - \frac{w^\mu}{c^2} \frac{\text{đ}\phi}{d\tau} \qquad \textit{with} \qquad f^0 = w^0 P/c \qquad (9)
$$

This system of equations, Eqs. (8) and (9), represents a complete set of equations to determine the motion of the charge. It has to be pointed out that **đ** does not represent an exact differential as it happens with the heat in Thermodynamics. This point represents a correction to Hammond theory. Indeed, we will see that $\phi = \phi(x_\mu, w_\mu)$ and therefore, there is a difference between $\text{đ}\phi/d\tau$ and $d\phi d\tau$; that is:

$$
\frac{\text{đ}\phi}{d\tau} = \frac{\partial\phi}{\partial x_\mu} w_\mu \qquad \textit{and} \qquad \frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial x_\mu} w_\mu + \frac{\partial\phi}{\partial w_\mu} a_\mu. \qquad (10)
$$

Physically, this is consistent with the fact that non exact differentials are always connected with no reversible processes as the radiation. The fact that ϕ has to be deduced for each applied force to the charge implies that there is no a H equation of motion but a H method to obtain an equation of motion for each case. Then,

$$
a^\mu = \frac{dw^\mu}{d\tau} = \frac{e}{mc} F^{\mu\sigma} w_\sigma + \frac{1}{m} \phi^{,\mu} - \frac{w^\mu}{c^2 m} \frac{\text{đ}\phi}{d\tau}. \qquad (11)
$$

Following Hammond [36] but including our correction, we arrive at:

$$P = \frac{\text{đ}\phi}{d\tau} = \frac{\partial\phi}{\partial x_\mu} w_\mu \qquad \textit{with} \qquad P = -\tau_o m a^2 = -\tau_o m a_\mu a^\mu. \tag{12}$$

Then,

$$\text{đ}\phi = P d\tau = -\tau_o m a_\mu a^\mu d\tau = -\tau_o m \frac{dw_\mu}{d\tau}\frac{dw^\mu}{d\tau} d\tau. \tag{13}$$

2.2.1. Constant electric field in the x^1 direction within Hammond theory

In order to analyze the constant electric field case in the x^1 direction and to be able to solve Eq. (11) it is necessary to make the following approximation (first order in τ_o): The Lorentz acceleration is taken to evaluate P; that is:

$$\begin{aligned} P &= -\tau_o m a_\mu a^\mu = -\tau_o m \left(\left(\frac{e}{cm}\right)^2 F_{\alpha\nu} w^\nu F^{\alpha\beta} w_\beta \right) \\ &= -\tau_o m \left(\frac{e}{cm}\right)^2 E^2 \left(-w_x w^x - w_0 w^0\right) \\ &= \tau_o m \frac{e^2}{m^2} E^2 = \tau_o \frac{e^2}{m} E^2. \end{aligned} \tag{14}$$

Therefore,

$$\text{đ}\phi = P d\tau = \tau_o \frac{e^2}{m} E^2 d\tau. \tag{15}$$

Knowing that,

$$d\tau = \frac{dt}{\gamma}, \tag{16}$$

we must have

$$\begin{aligned} \text{đ}\phi &= \frac{\partial\phi}{\partial x_\mu} dx_\mu = \frac{\partial\phi}{\partial x_0} dx_0 + \frac{\partial\phi}{\partial x_1} dx_1 \\ &= \phi^{,0} dx_0 + \phi^{,1} dx_1 \\ &= P d\tau = P\frac{dt}{\gamma} = \frac{P}{\gamma} dt = \frac{P}{\gamma c} d\left(ct\right) = \frac{P}{\gamma c} dx_0. \end{aligned} \tag{17}$$

By using Eqs. (13) y (17), we have

$$\frac{\text{đ}\phi}{d\tau} = P = \tau_o \frac{e^2}{m} E^2, \qquad \phi^{,0} = \frac{P}{\gamma c} \quad \textit{and} \quad \phi^{,1} = 0. \tag{18}$$

On the other hand,

$$\begin{aligned} w_\mu \left(\phi^{,\mu} - \frac{w^\mu}{c^2} \frac{\text{đ}\phi}{d\tau} \right) &= w_\mu \phi^{,\mu} - \frac{w_\mu w^\mu}{c^2} \frac{\text{đ}\phi}{d\tau} \\ &= w_\mu \frac{\partial\phi}{\partial x_\mu} - w_\mu \frac{\partial\phi}{\partial x_\mu} = 0. \end{aligned} \tag{19}$$

This result used in Eq. (11) permits to check the balance of energy. It has to be remembered that

$$\phi^{,0} = \frac{\partial\phi}{\partial x_0} = \frac{P}{\gamma c} = \tau_o \frac{e^2}{\gamma c m} E^2 = \tau_o \frac{e^2}{w^0 m} E^2, \tag{20}$$

and consequently $\phi = \phi(x^\mu, w^0)$ and in general $\phi = \phi(x^\mu, w^\mu)$. Had we used $d\phi$, we would have obtained

$$
\begin{aligned}
w_\mu \left(\phi^{,\mu} - \frac{w^\mu}{c^2}\frac{d\phi}{d\tau} \right) &= w_\mu \phi^{,\mu} - \frac{w_\mu w^\mu}{c^2}\frac{d\phi}{d\tau} \\
&= w_\mu \phi^{,\mu} - \frac{d\phi}{d\tau} \\
&= w_\mu \phi^{,\mu} - w_\mu \phi^{,\mu} - a_\mu \frac{\partial \phi}{\partial w_\mu} \\
&= -a_\mu \frac{\partial \phi}{\partial w_\mu} \neq 0.
\end{aligned}
\tag{21}
$$

Hence, the balance of energy would not be satisfied. Therefore, we must use $đ\phi$. Finally, the radiation reaction term depends on the trajectory as it is expected.

We are able to express the equations of motion for the constant electric field; that is:

$$
\frac{dw^0}{d\tau} = \frac{eE}{cm}w^1 + \frac{1}{m}\frac{P}{\gamma c} - \frac{w^0}{c^2 m}P \tag{22}
$$

We obtain

$$
c^2 m\dot{\gamma} = eEw^1 + \frac{P}{\gamma} - P\gamma, \tag{23}
$$

where $\dot{\gamma} = d\gamma/d\tau$.

For x^1, we arrive at

$$
\begin{aligned}
\frac{dw^1}{d\tau} &= \frac{e}{mc}F^{\mu\sigma}w_\sigma + \frac{1}{m}\phi^{,1} - \frac{1}{m}\frac{w^1}{c^2}\frac{\boldsymbol{đ}\phi}{d\tau} \\
\frac{dw^1}{d\tau} &= \frac{eE}{mc}w^0 - \tau_o w^1 \frac{e^2}{c^2 m^2}E^2.
\end{aligned}
\tag{24}
$$

That is, by using $\Omega = eE/mc$,

$$
\frac{dw^1}{d\tau} = \Omega w^0 - \tau_o \Omega^2 w^1. \tag{25}
$$

Let us propose

$$
w^\mu = u^\mu + \tau_o v^\mu. \tag{26}
$$

Therefore, Eq. (25) can be written as

$$
\frac{dw^1}{d\tau} = \frac{d\left(u^1 + \tau_o v^1\right)}{d\tau} = \Omega\left(u^0 + \tau_o v^0\right) - \tau_o \Omega^2 \left(u^1 + \tau_o v^1\right). \tag{27}
$$

On the other hand, developing the identity $w_\mu w^\mu = c^2$,

$$
\begin{aligned}
c^2 &= w_0 w^0 + w_1 w^1 \\
&= \left(u_0 + \tau_0 v_0\right)\left(u^0 + \tau_0 v^0\right) + \left(u_1 + \tau_0 v_1\right)\left(u^1 + \tau_0 v^1\right), \\
&= u_0 u^0 + u_1 u^1 + 2\tau_0 \left[u_0 v^0 + u_1 v^1\right] + \tau_0^2 \left(v_0 v^0 + v_1 v^1\right),
\end{aligned}
$$

By identifying the coefficients of τ_0 and τ_0^2, we obtain

$$
u_0 v^0 + u_1 v^1 = 0, \qquad v_0 v^0 + v_1 v^1 = 0. \tag{28}
$$

Therefore,

$$v^1 = -\frac{u_0 v^0}{u_1} = \frac{u^0 v^0}{u^1}, \tag{29}$$

$$(v_0)^2 = -v_1 v^1 = (v^1)^2. \tag{30}$$

These identities will be used later in order to deduce v^0 in terms of v^1, u^0 and u^1. Then, from Eq. (25), neglecting the terms of order τ_o^2, we have

$$\frac{d\left(u^1 + \tau_o v^1\right)}{d\tau} = \Omega\left(u^0 + \tau_o v^0\right) - \tau_o \Omega^2 \left(u^1 + \tau_o v^1\right). \tag{31}$$

Identifying the terms depending on τ_o or not, we obtain

$$\frac{du^1}{d\tau} = \Omega u^0 \qquad \textit{and} \qquad \frac{dv^1}{d\tau} = \Omega v^0 - \Omega^2 u^1. \tag{32}$$

Now, we need to obtain w^0. By using Eq. (22) and by substituting P from Eq. (18), we have

$$\frac{dw^0}{d\tau} = \Omega w^1 + \tau_o \Omega^2 \left(\frac{c}{\gamma} - w^0\right). \tag{33}$$

By using Eq. (26) into Ec. (33), with $w^0 = u^0 + \tau_o v^0$, we have

$$\begin{aligned} \frac{dw^0}{d\tau} = \frac{du^0 + \tau_o v^0}{d\tau} &= \Omega\left(u^1 + \tau_o v^1\right) \\ &\quad + \tau_o \Omega^2 \left(\frac{c^2}{u^0 + \tau_o v^0} - u^0 + \tau_o v^0\right) \\ \frac{du^0}{d\tau} + \tau_o \frac{dv^0}{d\tau} &= \Omega u^1 + \tau_o \Omega v^1 + \tau_o \Omega^2 \left(\frac{c^2}{u^0} - u^0\right). \end{aligned} \tag{34}$$

Therefore,

$$\frac{du^0}{d\tau} = \Omega u^1 \qquad \textit{and} \qquad \frac{dv^0}{d\tau} = \Omega v^1 + \Omega^2 \left(\frac{c^2}{u^0} - u^0\right). \tag{35}$$

From Eq. (32) and (35), we can obtain the solution for u^μ; that is:

$$\frac{du^0}{d\tau} = \Omega u^x \Rightarrow u^0 = c\cosh\Omega\tau \quad \textit{and} \quad \frac{du^x}{d\tau} = \Omega u^0 \Rightarrow u^1 = c\sinh\Omega\tau. \tag{36}$$

Therefore,

$$\frac{dv^1}{d\tau} - \Omega v^0 = -c\Omega^2 \sinh\Omega\tau. \tag{37}$$

Then, we need to express v^0 in order to solve the last equation. From Eq. (29), we have

$$v^0 = \frac{u^1}{u^0} v^1. \tag{38}$$

Then,

$$\frac{dv^1}{d\tau} - \Omega \frac{u^1}{u^0} v^1 = -c\Omega^2 \sinh\Omega\tau. \tag{39}$$

By using Eq. (36), we arrive at

$$\frac{dv^1}{d\tau} - \Omega \tanh(\Omega\tau)\, v^1 = -c\Omega^2 \sinh \Omega\tau. \tag{40}$$

The solution is

$$v^1 = c\tau_0 \Omega \cosh \Omega\tau \left(\Omega\tau - \ln\left(\frac{1 + e^{2\Omega\tau}}{2} \right) \right). \tag{41}$$

Finally, the solution for w^1 is:

$$\begin{aligned} w^1 &= c \sinh \Omega\tau + \tau_0 v^1 \\ &= c \sinh \Omega\tau + c\tau_0 \Omega \cosh \Omega\tau \left(\Omega\tau - \ln\left(\frac{1 + e^{2\Omega\tau}}{2} \right) \right) \end{aligned} \tag{42}$$

For solving w^0, instead of solving directly Eq. (35), from Eq. (38), we can deduce

$$v^0 = \frac{u^1}{u^0} v^1 = \tanh(\Omega\tau)\, v^1 = c\Omega \sinh \Omega\tau \left(\Omega\tau - \ln\left(\frac{1 + e^{2\Omega\tau}}{2} \right) \right) \tag{43}$$

Finally,

$$w^0 = c \cosh \Omega\tau + c\tau_0 \Omega \sinh \Omega\tau \left(\Omega\tau - \ln\left(\frac{1 + e^{2\Omega\tau}}{2} \right) \right). \tag{44}$$

These solutions are described in figures (1) and (2).

2.3. Comparison between the Solutions of the L and LL Equations and the Hammond Method for the Constant Electric Field

As we have noticed, the solutions of the L and LL equations for the constant electric field coincide. However, by using H method, in this case, the solution is different from the L and LL solutions since a loss of energy appears in the H one. This loss of energy which eliminates the constant electric field paradox can be seen in figures (1) and (2). Apparently, this effect given by using the H method justified the use of it to the detriment of the ideas of many authors that consider that the term in the LL equation due to the attached fields (the Scott term) is the responsible of eliminating the radiation term in the LL equation [12], [26], [20].

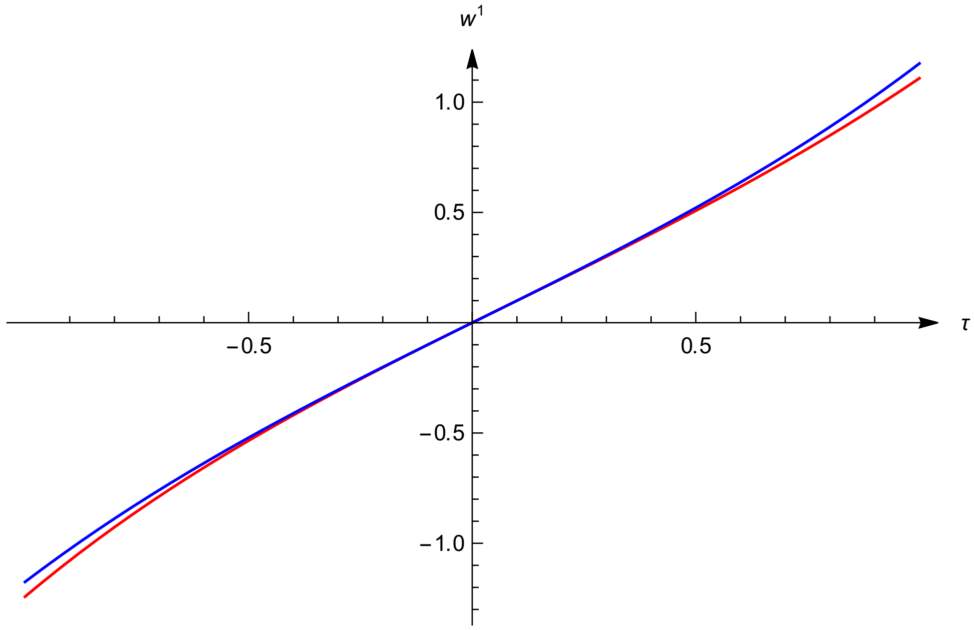

Figure 1. Comparison of w^1 between the L, the LL, and the H solutions for the constant electric field: L and LL in blue and H in red; $\Omega = 1$ and $\tau_0 = 0.1$ to show the effects.

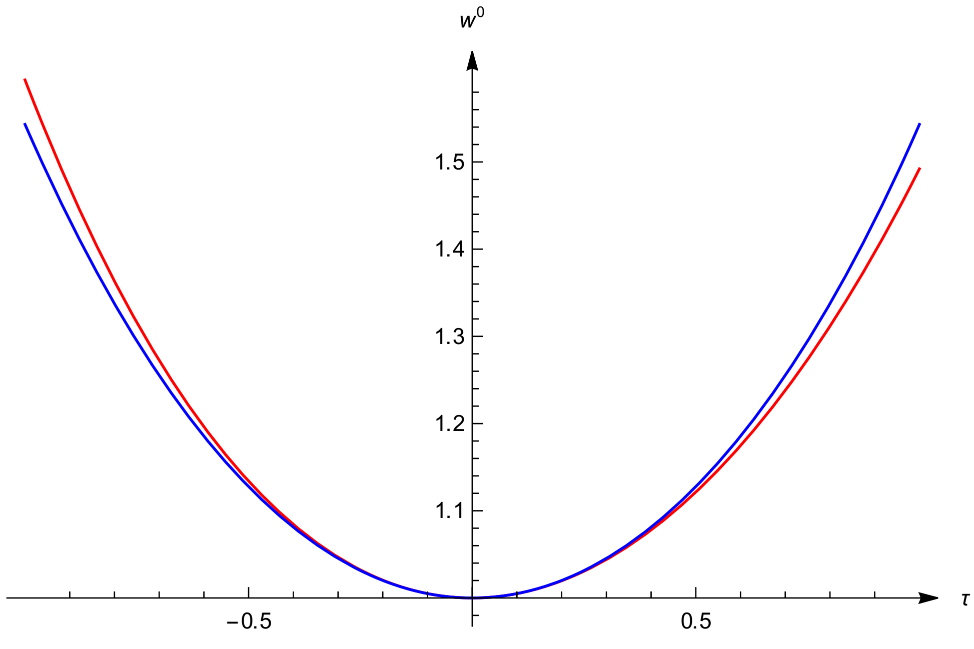

Figure 2. Comparison of w^0 between the L, the LL and the H solutions for the constant electric field: L and LL in blue and H in red; $\Omega = 1$ and $\tau_0 = 0.1$ to show the effects.

3. Step and Smooth Step Electric Field for the Lorentz, the Landau-Lifshitz Equations and the Hammond Method

The step electric field in reality does not represent a physical attainable field, as the constant field. However, the comparison between the solutions for the L and LL equations, and for the H method, for the step electric field, permits us to understand the differences between them. Of course, a smooth step electric field should be considered later to analyze a physical situation.

3.1. The Step Electric field for the Lorentz and Landau-Lifshitz equations and the Hammond method

Let us start by considering an electric field in the x^1 direction that behaves like a step function; that is:

$$E = \left\{ \begin{array}{ccc} 0 & for & \tau < 0 \\ E_o & for & \tau \geq 0 \end{array} \right\} = E_o H(\tau) . \tag{45}$$

3.1.1. The solutions for the Lorentz equation with a step electric field

The solutions for the L equation are simple and have to be obtained for two cases:
1° case, $\tau < 0$.
The solution is

$$w^1 = 0 \qquad and \qquad w^0 = c \tag{46}$$

2° case, $\tau \geq 0$.
The solution is

$$w^1 = c \sinh \Omega_0 \tau \qquad and \qquad w^0 = c \cosh \Omega_0 \tau, \tag{47}$$

with $\Omega_0 = eE/mc$. These solutions are described in figures (3) and (4).

3.1.2. The solutions for the Landau-Lifshitz equation with a step electric field

The component LL equations, Eq. (3), for the step electric field, Eq. (45), are: for w^1

$$\begin{aligned} \frac{dw^1}{d\tau} &= \Omega_0 H(\tau) w^0 \\ &+ \tau_o \Omega_0 \left[\delta(\tau) w^0 + H(\tau) \frac{dw^0}{d\tau} \right] \\ &+ \tau_o \Omega_0^2 H(\tau)^2 w^1, \end{aligned} \tag{48}$$

and for w^0,

$$\begin{aligned} \frac{dw^0}{d\tau} &= \Omega_0 H(\tau) w^1 \\ &+ \tau_o \Omega_0 \left[\delta(\tau) w^1 + H(\tau) \frac{dw^1}{d\tau} \right] \\ &+ \tau_o \Omega_0^2 H(\tau)^2 w^0. \end{aligned} \tag{49}$$

Let us propose a general solution of the type:

$$w^1 = c \sinh \Psi \qquad and \qquad w^0 = c \cosh \Psi, \tag{50}$$

where $\Psi = \Psi(\tau)$. Introducing Eq. (50) into Eqs. (48) and (49), we obtain:

$$\dot{\Psi} = \Omega_0 (H(\tau)) + \tau_o \Omega_0 (\delta(\tau)), \tag{51}$$

which coincides with the result found by Baylis and Huschilt [43] for the LL equation. After a simple integration, considering the initial conditions, we arrive to:

$$\Psi = \left\{ \begin{array}{ccc} 0 & \text{for} & \tau < 0 \\ \Omega_0 \tau + \Omega_0 \tau_o & \text{for} & \tau \geq 0 \end{array} \right\} \tag{52}$$

Therefore, we have two cases:

1° case, $\tau < 0$

The solutions are

$$w^1 = 0 \qquad and \qquad w^0 = c \tag{53}$$

$2\circ$ case, $\tau \geq 0$

The solutions are

$$w^1 = c \sinh\left(\Omega_0\left(\tau + \tau_0\right)\right) \qquad and \qquad w^0 = c \cosh\left(\Omega_0\left(\tau + \tau_0\right)\right) \tag{54}$$

These solutions are described in figures (3) and (4).

3.1.3. The solutions for the Hammond method with a step electric field

We must now obtain the motion of a charge within Hammond theory in the case of a step electric field in x^1 direction. Therefore, we use the electric field described in Eq. (45). Eq. (40) is still valid

$$\frac{dv^1}{d\tau} - \Omega \tanh\left(\Omega\tau\right) v^1 = -c\Omega^2 \sinh \Omega\tau, \tag{55}$$

but with a different Ω,

$$\Omega = \begin{cases} 0 & for \quad \tau < 0 \\ \frac{eE_o}{cm} = \Omega_0 & for \quad \tau \geq 0 \end{cases} \tag{56}$$

The problem can be divided in two cases:

1° case, $\tau < 0 \Rightarrow \Omega = 0$

Then, Eq. (40) can be written as

$$\frac{dv^1}{d\tau} = 0. \tag{57}$$

Hence,

$$\frac{dw^1}{d\tau} = 0 \Rightarrow w^1 = 0. \tag{58}$$

2° case, $\tau \geq 0 \Rightarrow \Omega = \frac{eE_o}{cm} = \Omega_0$

First,

$$u^1 = c \sinh \Omega_0 \tau \tag{59}$$

and for v^1

$$\frac{dv^1}{d\tau} - \Omega \tanh\left(\Omega\tau\right) v^1 = -c\Omega^2 \sinh \Omega\tau, \tag{60}$$

The solution for v^1 is:

$$v^1 = c\tau_0\Omega_0 \cosh \Omega_0\tau \left(\Omega_0\tau - \ln\left(\frac{1 + e^{2\Omega_0\tau}}{2}\right)\right). \tag{61}$$

Then,

$$\begin{aligned} w^1 &= c \sinh \Omega_0\tau + \tau_0 v^1 \\ &= c \sinh \Omega_0\tau + c\tau_0\Omega_0 \cosh \Omega_0\tau \left(\Omega_0\tau - \ln\left(\frac{1 + e^{2\Omega_0\tau}}{2}\right)\right). \end{aligned} \tag{62}$$

Following the same method, we obtain

$$w^0 = c \cosh \Omega_0\tau + c\tau_0\Omega_0 \sinh \Omega_0\tau \left(\Omega_0\tau - \ln\left(\frac{1 + e^{2\Omega_0\tau}}{2}\right)\right). \tag{63}$$

These solutions are described in figures (3) and (4).

3.1.4. Comparison between the solutions of the L and LL equations and the Hammond method for the step electric field

In figures (3) and (4), it is shown that there is a loss of energy in the H method which corresponds to the Hammond's aims about the radiation reaction force. On the contrary, the solution of the LL equation presents a suddenly gain of energy. These two different results represent one of the controversies between the LL equation and the H method. However, being the step electric field a nonphysical field due to its discontinuity, the discussion about which is the best option is somewhat disqualified. Accordingly, we need to compare the results by using a physical field. This field can be expressed by a smooth step electric field which is physically possible.

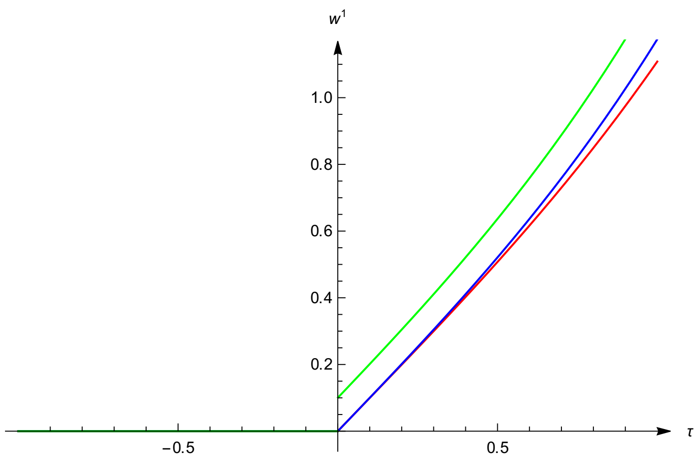

Figure 3. Comparison of w^1 between the L and the LL, and the H solutions for the step electric field: L in blue, LL in green and H in red; $\Omega_0 = 1$ and $\tau_0 = 0.1$ to show the effects.

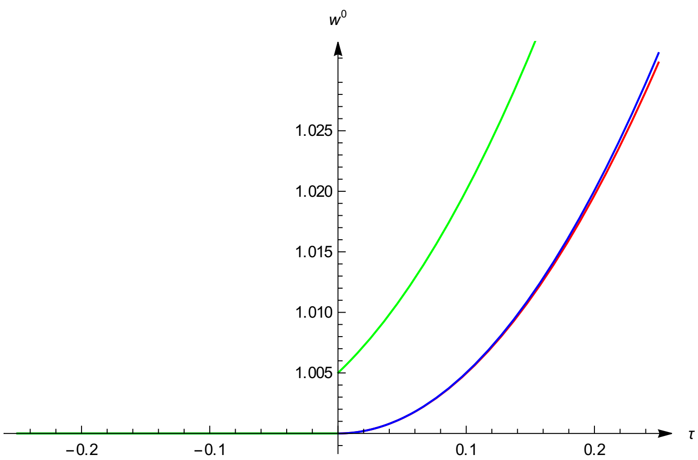

Figure 4. Comparison of w^0 between the L and the LL, and the H solutions for the step electric field: L in blue, LL in green and H in red; $\Omega_0 = 1$ and $\tau_0 = 0.1$ to show the effects.

3.2. The Smooth Step Electric Field for the Lorentz and Landau-Lifshitz Equations and the Hammond Method

Let us use a smooth step electric field which represents a physical situation and it can be represented by

$$\mathcal{E}\mathbf{E}, \tag{64}$$

where $\mathbf{E}$ is a constant vector in the x^1 direction related with the intensity of the electric field and $\mathcal{E}$ is

$$\mathcal{E} = \frac{1}{2}\left(1 + \tanh\left(\frac{\tau}{T}\right)\right). \tag{65}$$

Notice that when $T \to 0$, the step electric field is obtained as a limit. However, in this case we will consider $T = 1$ in order to use a clearly smooth step electric field (see figure (5)).

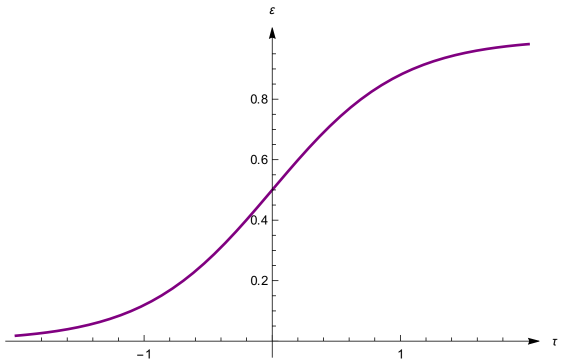

Figure 5. Smooth step function $\mathcal{E}$ with $T = 1$

.

3.2.1. The solutions for the Lorentz equation with a smooth step electric field

The equations of motion by using the L equation in the case of a smooth step electric field and putting $\Omega = \frac{eE}{mc}$ are:

$$\begin{aligned} \frac{dw^0}{d\tau} &= \Omega\mathcal{E}w^1, \\ \frac{dw^1}{d\tau} &= \Omega\mathcal{E}w^0. \end{aligned} \tag{66}$$

By following the algebra done by Hammond [33] and by considering the initial conditions as $x^1(-\infty) = 0$ and $w^1(-\infty) = 0$, the results are

$$w^0 = \frac{c}{2\sqrt{2}} \frac{e^{-\Omega\tau/2}\left(e^{2\Omega\tau} + 2\right)}{\sqrt{\cosh \Omega\tau}}, \tag{67}$$

and

$$w^1 = \frac{c}{2\sqrt{2}} \frac{e^{\frac{3}{2}\Omega\tau}}{\sqrt{\cosh \Omega\tau}}. \tag{68}$$

These solutions are described in figure (6).

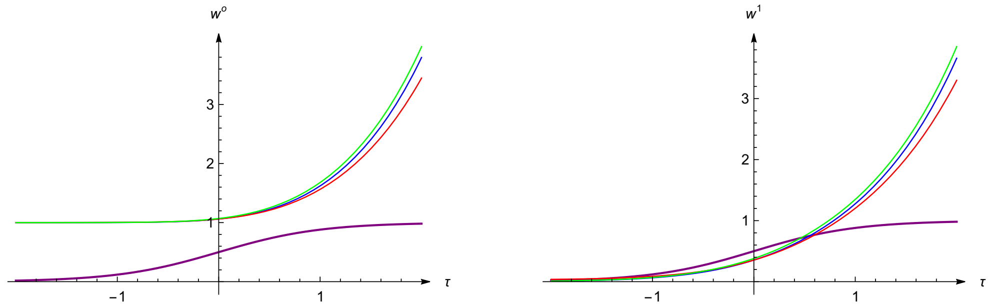

Figure 6. w^0 and w^1 for the smooth step electric field are shown with $T = 1$: L in blue, LL in green and H in red; the smooth step electric field is in violet; $\Omega_0 = 1$ and $\tau_0 = 0.1$ to show the effects.

3.2.2. The solutions for the Landau-Lifshitz equation with a smooth step electric field

The LL equation in the case of a smooth step electric field is (with $\Omega = \frac{eE}{mc}$)

$$\begin{aligned} \frac{dw^0}{d\tau} &= \Omega \mathcal{E} w^1 + \tau_0 \Omega \frac{d\mathcal{E}}{d\tau} w^1, \\ \frac{dw^1}{d\tau} &= \Omega \mathcal{E} w^0 + \tau_0 \Omega \frac{d\mathcal{E}}{d\tau} w^0, \end{aligned} \tag{69}$$

We solve these equations by using Wolfram Mathematica and the solutions are described in figure (6).

3.2.3. The solutions by using the Hammond method with a smooth step electric field

By applying the Hammond method [33] which consists o putting

$$w^\mu = u^\mu + \tau_o v^\mu, \tag{70}$$

Hammond obtained that u^μ coincides with the solution of the L equation, Eqs. (67) and (68), as it is expected. The solutions for v^0 and v^1 are

$$v^0 = \frac{c\Omega e^{3\Omega\tau/2}}{8\left(e^{2\Omega\tau}+1\right)^{3/2}\sqrt{\cosh \Omega\tau}} \left[\begin{array}{c} e^{\Omega\tau/2}\left(e^{2\Omega\tau}+1\right)\sqrt{\cosh\Omega\tau}\left(1-\ln(4)\right) \\ +\sqrt{2}\left(\left(e^{2\Omega\tau}+1\right)\ln\left(e^{2\Omega\tau}+1\right)-1\right)\sqrt{e^{2\Omega\tau}+1} \end{array} \right], \tag{71}$$

and

$$v^1 = \frac{c\Omega e^{-\Omega\tau/2}\left(e^{2\Omega\tau}+2\right)}{8\left(e^{2\tau}+1\right)^{3/2}\sqrt{\cosh \tau}} \left[\begin{array}{c} e^{\Omega\tau/2}\left(e^{2\Omega\tau}+1\right)\sqrt{\cosh\Omega\tau}\left(1-\ln(4)\right) \\ +\sqrt{2}\left(\left(e^{2\Omega\tau}+1\right)\ln\left(e^{2\Omega\tau}+1\right)-1\right)\sqrt{e^{2\Omega\tau}+1} \end{array} \right]. \tag{72}$$

These solutions are described in figure (6).

3.2.4. Comparison between the solutions of the Lorentz and Landau-Lifshitz equations and the Hammond method for the smooth step electric field

In figures (6), it is shown that there is also a loss of energy in the H method which corresponds to the Hammond's aims about the radiation reaction force as happens in the case of the step electric field. On the contrary, the solution of the LL equation presents a gain of energy but, unlike the previous case, there is no discontinuity. This allows us to see the discrepancy between the LL equation and H method. The loss and gain of energies depending on the chosen equation exemplifies the differences between both theories. This will be discussed after analyzing the electromagnetic pulse case for both theories.

4. The Electromagnetic Pulse by using the Lorentz and Landau-Lifshitz Equations, and the Hammond Method

The study of the electromagnetic waves in electrodynamics is fundamental. Above all, many physical effects are explained by analyzing the motion of charged particles subjected to an electromagnetic pulse. Let us now consider a polarized electromagnetic pulse, in the x direction ($x = x^1$); that is:

$$\overrightarrow{E} = Eh\hat{x}, \tag{73}$$

where E is a constant and $h = h(z - t/c)$. The corresponding magnetic field is:

$$\overrightarrow{B} = Eh\hat{y}, \tag{74}$$

Following Hammond [33], by making the following scale transformations $x^\mu \to x^\mu/L$, $t \to tc/L$ with $L = \lambda/2\pi$ (note that L and λ are not related with the wavelength of the pulse). We will call this scale transformation as the dimensionless convention. h can be expressed as

$$h = \frac{1}{w} e^{-((z-t)/w)^2} \cos\left(\Omega\left(z - t\right)\right). \tag{75}$$

where w is a dimensionless parameter that controls the width of the Gaussian and Ω is a dimensionless parameter controlling the frequency. Notice that Hammond [35] chose $w = 2\pi/\Omega$ in order to maintain an envelope containing a few wavelengths. We will use other parameters by putting $\lambda = 5$, $\Omega = 0.1$, $w = 2\lambda/\Omega$, with an average intensity $I = 10^{22}$ Wcm^{-2} where $E = (8\pi I/c)^{-1/2}/w$. All the solutions described in figures (7), (8), (9), (10), (11), (12) and (13) are described by using these ultrahigh intensity electromagnetic pulse.

4.1. The Electromagnetic Pulse by using the Lorentz Equation

Let us begin by solving the L equation for such electromagnetic pulse and the above dimensionless convention. Let us put $a = \frac{eE}{mc}$, then the L equation can be written as

$$\begin{aligned} \frac{dw^0}{d\tau} &= ahw^1, \\ \frac{dw^1}{d\tau} &= ah\left(w^0 - w^3\right), \\ \frac{dw^2}{d\tau} &= 0, \\ \frac{dw^3}{d\tau} &= ahw^1. \end{aligned} \tag{76}$$

Then

$$\frac{dw^0}{d\tau} = \frac{dw^3}{d\tau} \tag{77}$$

Integrating from $\tau = -\infty$ to τ, we arrive at

$$w^0\left(\tau\right) - w^0\left(-\infty\right) = w^3\left(\tau\right) - w^3\left(-\infty\right). \tag{78}$$

By putting the initial conditions as $w^0(-\infty) = 1$ and $w^1(-\infty) = w^2(-\infty) = w^3(-\infty) = 0$, we arrive at

$$w^0\left(\tau\right) = 1 + w^3\left(\tau\right). \tag{79}$$

Integrating from $\tau = -\infty$ to τ and using the initial conditions, $x^0(-\infty) = -\infty = \tau(-\infty)$ and $x^1(-\infty) = x^2(-\infty) = x^3(-\infty) = 0$, we obtain

$$\tau = t - z, \tag{80}$$

which represents an important result. Then, we can write

$$h = \frac{1}{w} e^{-(\tau/w)^2} \cos(\Omega\tau)\,. \tag{81}$$

The solutions are

$$\begin{aligned} w^0 &= 1 + a^2\mathcal{E}^2, \qquad w^1 = a\mathcal{E}, \\ w^2 &= 0, \qquad w^3 = a^2\mathcal{E}^2, \end{aligned} \tag{82}$$

where,

$$\mathcal{E}(\tau) = \int_{-\infty}^{\tau/w} e^{-\zeta^2} \cos(2\Lambda\zeta)\, d\zeta, \tag{83}$$

with $\zeta = \tau/w$ and $2\Lambda = w\Omega$. The solutions of these equations are described in figures (7), (8), (9), (10) and (11).

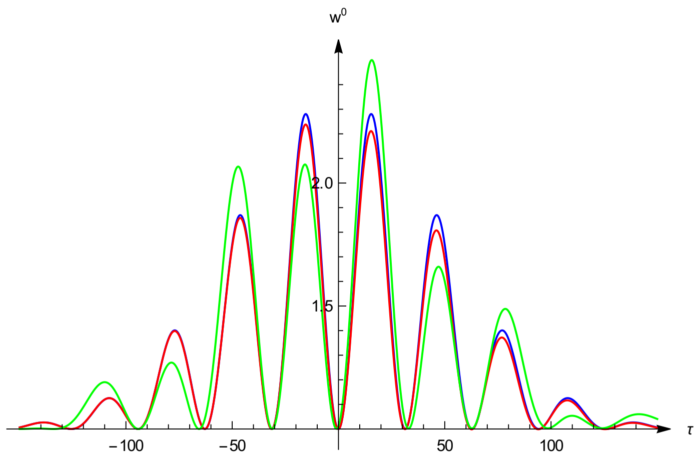

Figure 7. w^0 for the electromagnetic pulse: L in blue, HLD in red and LL in green.

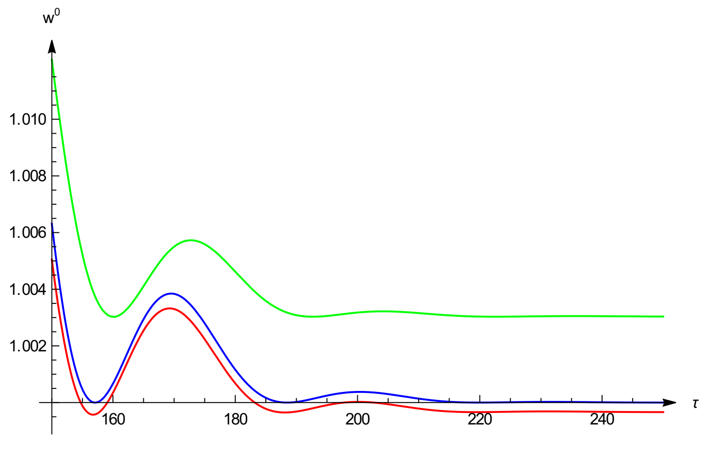

Figure 8. Close-up of the interval (150, 250), w^0 for the electromagnetic pulse: L in blue, HLD in red and LL in green.

4.2. The Electromagnetic Pulse by using Landau-Lifshitz Equation
For the same electromagnetic pulse with the dimensionless convention and using as a good approximation Eq. (80), the LL equations are

$$
\begin{aligned}
\frac{dw^0}{d\tau} &= a\left(h+\tau_0\dot{h}\right)w^1+\tau_0 a^2h^2(w^0-w^3)(1-w^0(w^0-w^1)),\\
\frac{dw^1}{d\tau} &= a\left(h+\tau_0\dot{h}\right)(w^o-w^3)-\tau_0 a^2h^2(w^0-w^3)^2w^1,\\
\frac{dw^2}{d\tau} &= -\tau_0 a^2h^2(w^0-w^3)^2w^2,\\
\frac{dw^3}{d\tau} &= a\left(h+\tau_0\dot{h}\right)w^1+\tau_0 a^2h^2(w^0-w^3)(1-w^3(w^0-w^1)).
\end{aligned}
\tag{84}
$$

The solutions of these equations are obtained by using Wolfram Mathematica and they are described in figures (7), (8), (9), (10), (11) and (13).

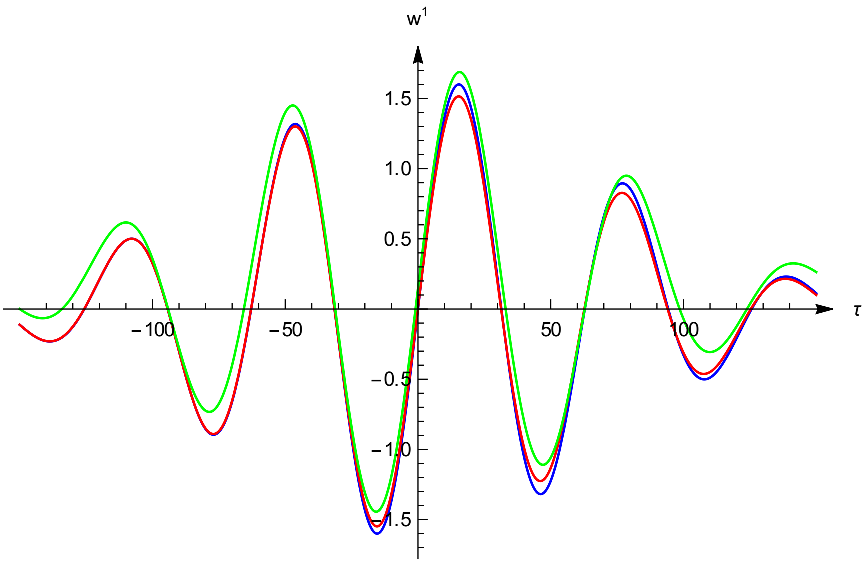

Figure 9. w^1 for the electromagnetic pulse: L in blue, HLD in red and LL in green.

4.3. The Electromagnetic Pulse by using the Hammond Method
For the same pulse, Hammond first use a variation of the LD equation [32] that we will call it the Hammond-Lorentz-Dirac equation [HLD] and consists of using the following equation:

$$\frac{dw^\sigma}{d\tau}=aF^{\sigma\mu}w_\mu+\tau_o\left[\frac{d}{d\tau}\left(aF^{\sigma\mu}w_\mu\right)+\left(\dot{w}^\mu\dot{w}_\mu\right)w^\sigma\right]+\mathcal{O}(\tau_o^2), \tag{85}$$

Notice that this equation does not accomplish the balance of energy and a a consequence some nonphysical results at first order in τ_o can appear. This can be noticed in figure 8 where the energy is less than c (less than one in the figure due to the dimensionless convention) in some parts. By using Eq. (80) and the dimensionless convention, the HLD equations are:

$$
\begin{aligned}
\frac{dw^0}{d\tau} &= ahw^1+\tau_o a^2\dot{h}\mathcal{E}-\tau_o\frac{a^4h^2\mathcal{E}^2}{2},\\
\frac{dw^1}{d\tau} &= ah(w^0-w^3)+\tau_o a\dot{h}-\tau_o a^3h^2\mathcal{E},
\end{aligned}
$$

$$\frac{dw^2}{d\tau} = 0, \tag{86}$$
$$\frac{dw^3}{d\tau} = ahw^1 + \tau_o a^2 \dot{h}\mathcal{E} + \tau_o a^2 h^2 - \tau_o \frac{a^4 h^2 \mathcal{E}^2}{2},$$

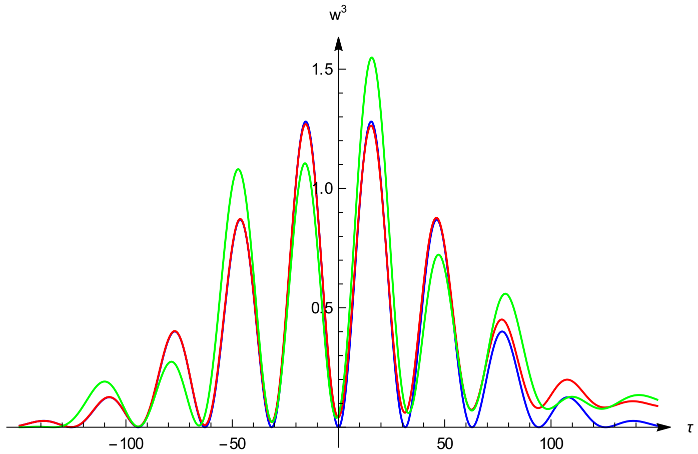

Figure 10. w^3 for the electromagnetic pulse: L in blue, HLD in red and LL in green. Notice that due to the initial conditions $w^2 = 0$.

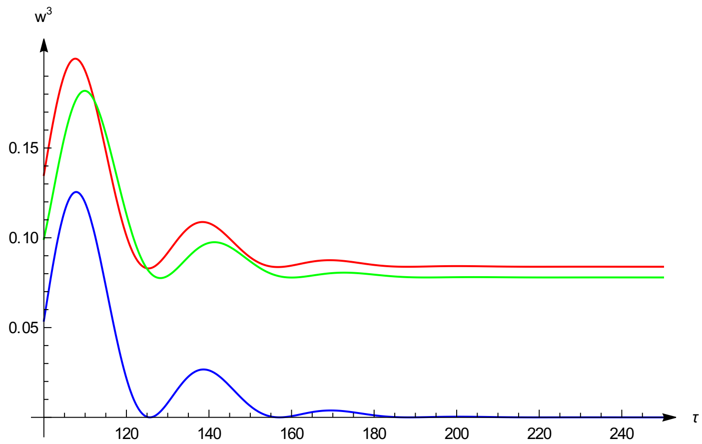

Figure 11. Close-up of the interval (150,250), w^3 for the electromagnetic pulse: L in blue, HLD in red and LL in green.

4.4. Hammond Approximation for an Ultrahigh Intensity Electromagnetic Pulse

Finally, Hammond [35] applied his method for the same pulse using Eq.(80) and the dimensionless convention but making the following approximation for an ultrahigh intensity electromagnetic pulse,

$$\phi^{,\mu} << \frac{w^\mu}{c^2}\frac{\mathbf{đ}\phi}{d\tau}. \tag{87}$$

Hammond obtain an equation by using Eq. (8) but with

$$f^\mu = -\frac{w^\mu}{c^2}\frac{\mathbf{đ}\phi}{d\tau}, \tag{88}$$

since he consider that

$$\phi^{,\mu} \simeq 0. \tag{89}$$

The equation of motion turns to be

$$\begin{aligned}
\frac{dw^0}{d\tau} &= aF^{0\mu}w_\mu + \tau_o \left(\dot{w}^\mu \dot{w}_\mu\right) w^0, \\
\frac{dw^i}{d\tau} &= aF^{i\mu}w_\mu - \frac{w^i}{c^2}\frac{\mathbf{d}\phi}{d\tau},
\end{aligned} \tag{90}$$

which can be called the Hammond approximated equation for ultrahigh intensity electromagnetic pulses [HA]. This is similar to the assumption done by Shen [40] for the LD equation by neglecting the Schott term. Nevertheless, although he obtained a similar result to the one obtained by using the HLD method, Hammond's results show differences with what he gets with the LL equation and the EFO equation (see figure 2 in reference [35]. The most important difference in the results obtained by Hammond [35] is the final z component of the velocity. Moreover, due to the unphysical runaways solutions, Hammond claimed that for this reason alone one may conclude that the LAD equation is incorrect, but it was shown that this arises from a deeper problem, explicitly, nonconservation of energy. Since the LL and EFO equations may be derived from the LAD equation, we should not expect them to be exact. Contrary to what Hammond [35] assures, those that fulfill the conservation of energy are the L, LL and EFO equations because they are based in Dirac's assumption of the conservation of the energy, where the Schott term appears in a natural way and those that do not comply with the conservation of energy are the HLD and HA equations. Moreover, the energy balance is accomplished by the L, the LL and the EFO equations and the H method. But in the cases of the HLD and the HA the energy balance is not accomplished as has been shown graphically in figure (8), Indeed, in general, it can be seen that the LDH and HA equations do not satisfy the energy balance simply multiplying them by w_μ. However, as we can see, our results obtained by using the LL equation, figure (11), shows that the difference are not such. Hammond managed the LL equation to obtain the numerical solution of the LL equation in the case of the ultrahigh intensity electromagnetic pulse in his article [35] but our result is obtained by directly using the LL equation and accordingly, the result is similar to LDH and HA one's.

Let us describe how Hammond solve the LL equation. He starts by neglecting the corresponding Schott term in the LL equation such that he obtains the following Landau-Lifshitz-Hammond equation for an ultrahigh intensity electromagnetic pulse [LLH]; that is:

$$\frac{dw^\mu}{d\tau} = F^{\mu\sigma}v_\sigma + \tau_o\left(\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau}\right)w^\mu, \tag{91}$$

and he used the Eqs. (73), (74) y (75), with $a = \frac{eEh}{mc}$. By using Eq. (80) and the dimensionless convention Eq. (91) maybe described by

$$\begin{aligned}
\frac{dw_0}{d\tau} &= ahw^1 + \tau_o\left(\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau}\right)w^0, \\
\frac{dw^1}{d\tau} &= ah\left(w^0 - w^3\right) + \tau_o\left(\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau}\right)w^1, \\
\frac{dw^2}{d\tau} &= 0 + \tau_o\left(\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau}\right)w^2, \\
\frac{dw^3}{d\tau} &= ahw_x + \tau_o\left(\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau}\right)w^3.
\end{aligned} \tag{92}$$

In order to solve this equation, it is necessary to use the following approximation,

$$\frac{dw^\mu}{d\tau}\frac{dw_\mu}{d\tau} = \frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}, \tag{93}$$

where $w^{\mu} = u^{\mu} + \tau_o v^{\mu}$ and by using Eq. (92), we obtain

$$\frac{dw^{\mu}}{d\tau}\frac{dw_{\mu}}{d\tau} = a^2h^2\left(w^1\right)^2 - a^2h^2(w^0 - w^3)^2 - a^2h^2\left(w^1\right)^2. \tag{94}$$

Simplifying,

$$\frac{dw^{\mu}}{d\tau}\frac{dw_{\mu}}{d\tau} = -a^2h^2(w^0 - w^1)^2, \tag{95}$$

By using Eqs (82) and (83), we have

$$\frac{dw^{\mu}}{d\tau}\frac{dw_{\mu}}{d\tau} = -a^2h^2\left(1 + \frac{1}{2}a^2h^2 - \frac{1}{2}a^2h^2\right)^2 = -a^2h^2, \tag{96}$$

and substituting in Eq. (92), we obtain

$$\begin{aligned}
\frac{dw_0}{d\tau} &= ahw_x - \tau_o a^2h^2w^0, \\
\frac{dw_x}{d\tau} &= ah\left(w_0 - w_z\right) - \tau_o a^2h^2w^1, \\
\frac{dw_y}{d\tau} &= -\tau_o a^2h^2w^2, \\
\frac{dw_z}{d\tau} &= ahw_x - \tau_o a^2h^2w^3.
\end{aligned} \tag{97}$$

The solutions of these equations are shown in figures (12) and (13), and it can be seen that it corresponds to the Hammond solution of the LL equation in reference [35] which does not corresponds to the real LL solution.

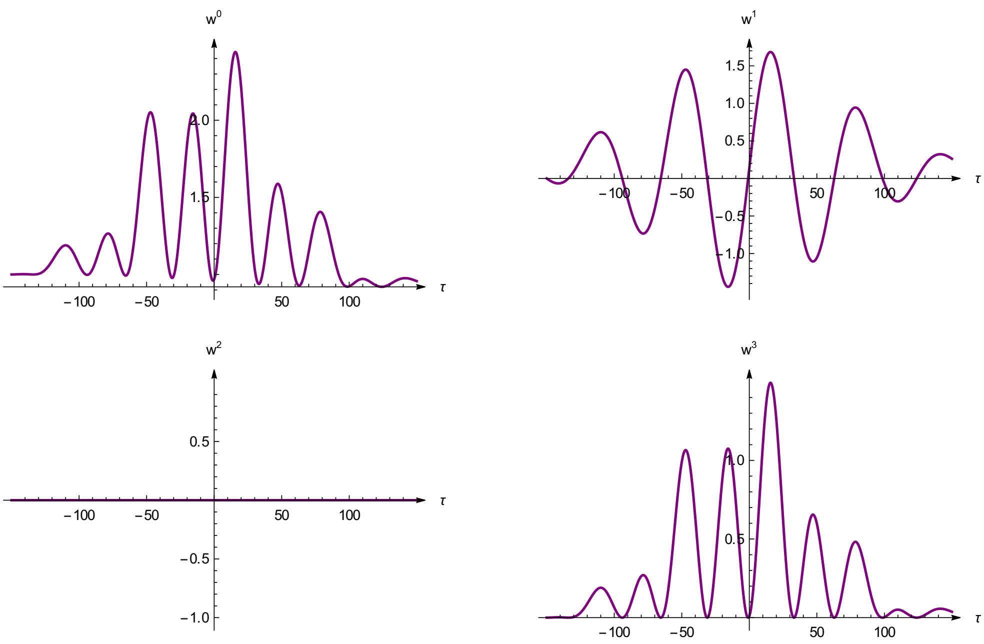

Figure 12. The solutions of w^{μ} for an ultrahigh intensity electromagnetic pulse by using the LLH equation.

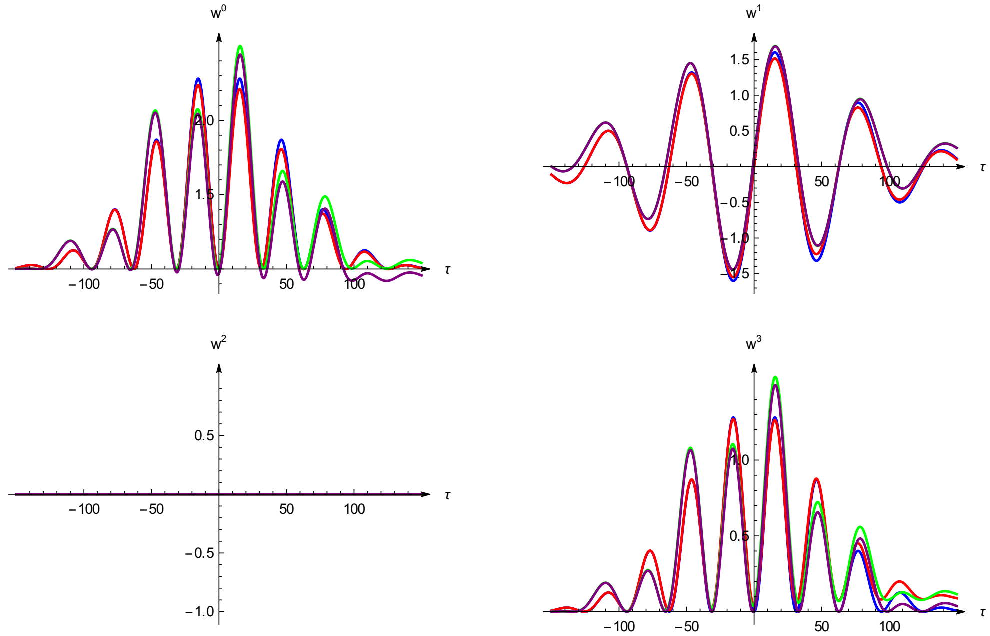

Figure 13. Comparison of the solutions w^μ for an ultrahigh intensity electromagnetic pulse for the different equations; L in blue, HLD and HA in red, LL in green and LLH in violet.

5. Discussion

Hammond [35] compared the EFO, the LL and HA (LDH is similar) results obtaining that for ultrahigh intensity pulses it is possible to experimentally measure the gain of net energy predicted by the Hammond theory described by analyzing the final w^3 components of each case (see figure (2) in [35]). It has to be noticed that the Lawson-Woodward theorem states: if radiation reaction is excluded, the particle gains no net momentum from the pulse. This happens when we consider the L solution. He claims that the gain of energy just happens when the LDH and the HA equations are considered. However, it has to be highlighted that the LL solutions (obtained by directly solving the LL equation) are very similar to LDH and HA solutions and there is also a gain of energy in z direction. However, the LDH and the HA equations do not satisfied the balance of energy and they did not come from a conservation of energy theories, neither the Hammond method unlike the LD, the LL and the EFO equations. Hammond did not obtain a correct descriptions of the LL and EFO solutions by using the LLH and a similar variation of the EFO by neglecting the Schott term.

For the constant magnetic field, it has been proved that the decay time and trajectories are similar for the Hammond theory [35] and the LL [38]; that is: $t_{decay} \propto 1/\tau_o w^2$. For the central field case, it has been shown that the Hammond method and the LL equation are equivalent [39].

6. Concluding Remarks

We can conclude that the big difference between Hammond method and the LL equation consists of a gain of energy that is sometimes present in the LL equation as happens in the cases of the constant electric field and of the step or smooth step electric field. However, it is all based on Hammond disregarding for the origin of the Schott term which comes from the field itself generated in the vicinity of the charged particle (the attached fields). In truth, this term causes that in some cases the particle gains kinetic energy but always keeping the energy balance. Do not forget that the conservation of the energy must consider the energy of the particle and the energy of the field. The balance of energy must be accomplished in each equation of motion and it only means that the energy generated is equal to the power, $ma^\mu w_\mu = F^\mu w_\mu = 0$, which are not satisfied in the LDH, the HA and LLH equations

The apparent constant force paradox is explained by other authors by noticing that the radiation exits at the infinity; that is, the energy radiated to infinity is taken from the attached fields (the Schott term or the acceleration energy) and consequently even if the total radiation term in the equation of motion vanishes, the radiation to the infinity (the irreversible emission of radiation) exists. Moreover, by using similar arguments, DeWitt and Brehme explain this phenomenon in his generalization to General Relativity of the damping term [20].

Moreover, a Landau-Lifshitz-like equation [22] in General Relativity has been proposed supporting the validity of the Landau-Lifshitz equation in Special Relativity. Finally, and perhaps the most important argument to support the LL equation has been done by Krivitskiĭ et al [25] by showing that the radiation reaction term represents an average radiation reaction force in Quantum Electrodynamics.

Finally, we can conclude that the LL equation of motion for a spin-less point-like charged particle represents the best proposal to describe the motion of such particles in Classical Electrodynamics.

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