Electromagnetism and spacetime geometry

Jussi Lindgren, Aalto University School of Science
Jukka Liukkonen, Radiation and Nuclear Safety Authority STUK

13.8.2020 IARD2020- The 12th Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields

- In our previous work [1] we have showed how coordinate-invariant (stochastic) optimization leads to the key equations of Quantum Mechanics.
- In the appendix of this presentation we show that using the volume form $\sqrt{g} d^4 x$ is compatible with General Relativity.
- This presentation shows first how the classical electromagnetic Lagrangian can be understood through the scalar curvature of the manifold. As is well-known, the Einstein field equations are then a stationarity condition for the metric tensor when looking a critical point for the total scalar curvature in a coordinate-invariant manner.
- Therefore the field equations of electromagnetism are a special case of the Einstein field equations.

Aims and the physical and mathematical framework

- How to understand electromagnetism from a purely geometrical perspective: Even though one can derive the field equations of electromagnetism from an ad hoc Lagrangian, the authors feel that there has not been a satisfying model on how to link electromagnetism and the theory of General Relativity.
- There have been many attempts to explain electromagnetic phenomena from relativity. Attempts by Einstein, Eddington, Weyl and Schrödinger are just an example of this approach. Geometrodynamics and the theories by Rainich [2], Misner and Wheeler [3] are of course important in this respect.

- Einstein himself was of the view that "A theory in which the gravitational field and the electromagnetic field do not enter as logically different structures would be much preferable." [4].
- How to connect electromagnetism with the canonical formulation of General Relativity?
- The mathematical framework is pseudo-Riemannian geometry.
- Only the canonical Levi-Civita connection is needed.

The technical framework

- Suppose we consider a pseudo-Riemannian manifold (the 4-dimensional spacetime) (M, g) with a torsionless and metrical connection ∇.
- We assume that the symmetric metric is given by the representation $g_{\mu\nu} = A_\mu A_\nu$ (tensor product).
- We can then start to impose some desirable features for the tensor A_μ in order to impose some desirable features for the metric itself.
- For didactical reasons, it could be useful to think of the tensor A_μ as a "vector field".

The optimization problem

If one thinks of the tensor A_μ as a "vector field", we could look for a vector field, which would be optimal in some sense. We proceed in this way, and we want to minimize the "rotation" of the vector field over the spacetime, whilst also we want the vector field to "travel along the level sets of divergence". Therefore, we look for a critical point for the following functional:

$$\int_M \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^\mu \nabla_\mu (\nabla^\nu A_\nu) \right) \sqrt{g} d^4 x. \qquad (1)$$

The invariant volume form ensures coordinate invariance, when integrating over the manifold. The rotating part of the above-defined cost functional is due to the tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The physical intuition for the latter integrand is the idea that the vector field should be orthogonal to the gradient of the divergence, i.e. parallel to the level sets of divergence.

We now show how the above "cost minimization" program is actually with mild assumptions the Einstein-Hilbert action. The first integrand is only the gradient energy of A_μ if the gradient tensor $\nabla_\nu A_\mu$ is antisymmetric:

$$F^{\mu\nu} F_{\mu\nu} = 4\nabla^\mu A^\nu \nabla_\mu A_\nu \tag{2}$$

Using Green's First Identity (with vanishing boundary terms), we have $\int_M \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \sqrt{g} d^4x = \int_M -A_\mu \Box A^\mu \sqrt{g} d^4x$.
Where we have the covariant d'Alembertian $\Box = \nabla^\mu \nabla_\mu$.

Consider now the covariant derivative of the tensor $F_{\mu\nu}$:

$$\nabla_\sigma F_{\mu\nu} = \nabla_\sigma \nabla_\mu A_\nu - \nabla_\sigma \nabla_\nu A_\mu. \tag{3}$$

Use the Ricci identity, which is essentially the definition for the Riemann-Christoffel curvature tensor:

$$\nabla_\sigma \nabla_\mu A_\nu = \nabla_\mu \nabla_\sigma A_\nu + R^\lambda_{\nu\sigma\mu} A_\lambda. \tag{4}$$

Substitute into the covariant derivative of the tensor $F_{\mu\nu}$:

$$\nabla_\sigma F_{\mu\nu} = \nabla_\mu \nabla_\sigma A_\nu + R^\lambda_{\nu\sigma\mu} A_\lambda - \nabla_\sigma \nabla_\nu A_\mu. \tag{5}$$

Next, we multiply equation 5 with the contravariant metric tensor $g^{\mu\nu}$:

$$\nabla_\sigma g^{\mu\nu} F_{\mu\nu} = \nabla^\nu \nabla_\sigma A_\nu + R^\lambda_\sigma A_\lambda - \nabla_\sigma \nabla^\mu A_\mu = 0. \tag{6}$$

The equation must be equal to zero, as the metric tensor is symmetric and the tensor $F_{\mu\nu}$ is antisymmetric. We also have made use of the metric compatibility of the Levi-Civita connection.

Contract with $\nu = \sigma$ and we have

$$\nabla^{\sigma}\nabla_{\sigma}A_{\sigma} + R^{\lambda}_{\sigma}A_{\lambda} - \nabla_{\sigma}\nabla^{\mu}A_{\mu} = 0. \tag{7}$$

For convenience, raise an index by multiplying with $g^{\sigma\nu}$

$$\Box A^{\nu} + R^{\nu\lambda}A_{\lambda} - \nabla^{\nu}\nabla^{\mu}A_{\mu} = 0. \tag{8}$$

Finally, multiply with A_{ν} (remember that $g_{\mu\nu} = A_{\mu}A_{\nu}$)

$$R = A_{\nu}\nabla^{\nu}\nabla^{\mu}A_{\mu} - A_{\nu}\Box A^{\nu}. \tag{9}$$

Which defines the Ricci curvature in terms of the A_{μ} and its covariant derivatives.

The Einstein-Hilbert Action

Comparing equation 1 we see that

$$\int_M \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^{\mu} \nabla_{\mu} (\nabla^{\nu} A_{\nu}) \right) \sqrt{g} d^4 x = \int_M R \sqrt{g} d^4 x \quad (10)$$

which is just the Einstein-Hilbert Action.

The Einstein field equations

- We show in the appendix how the Einstein field equations in vacuum are the stationarity condition for the functional above and that they are invariant with respect to the sign of the metric determinant.
- Obviously then, we need to consider the Einstein field equations within our context:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0. \quad (11)$$

Maxwell's equations from Einstein's equations

The Ricci tensor is obtained from equation 8, if we use a normalization $A_\mu A^\mu = 1$ (as in the case of four-velocity) and multiply equation 8 by A^λ:

$$A^\lambda \Box A^\nu + R^{\lambda\nu} - A^\lambda \nabla^\nu \nabla^\mu A_\mu = 0. \tag{12}$$

It is easy to see that if we have a metric which is Ricci-flat, the requirement becomes:

$$\Box A^\nu = \nabla^\nu (\nabla^\mu A_\mu) \tag{13}$$

Which are two of the four Maxwell's equations, if we identify A^ν as the electromagnetic four-potential and the term on the right side as the four-current density J^ν. The current is proportional to the gradient of the divergence of the four-potential. A Ricci-flat metric satisfies the Einstein field equation, and the metric $g_{\mu\nu} = A_\mu A_\nu$ is optimal.

- To recap, we have

$$\int_M \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A^\mu\nabla_\mu(\nabla^\nu A_\nu)\right)\sqrt{g}d^4x = \int_M R\sqrt{g}d^4x \tag{14}$$

  and the condition for Ricci-flatness:

$$\Box A^\nu = \nabla^\nu(\nabla^\mu A_\mu) = J^\nu, \tag{15}$$

  where we have the covariant d'Alembertian $\Box = \nabla^\mu\nabla_\mu$.
- The four-current J^ν can be coupled to the gradient of the divergence due to gauge invariance of the electromagnetic four-potential.

The Bianchi identity

The rest of the four Maxwell's equations are given by the algebraic Bianchi identity:

$$F_{[\lambda\mu;\nu]} = 0, \tag{16}$$

This cyclic permutation can be seen easily from the algebraic Bianchi identity which says that

$$R^{\lambda}_{\sigma\mu\nu} + R^{\lambda}_{\mu\nu\sigma} + R^{\lambda}_{\nu\sigma\mu} = 0 \tag{17}$$

where the semicolon refers to covariant differentiation. Substituting the definition of the Faraday tensor in the above and using the Ricci identity we end up with the algebraic Bianchi identity, which guarantees us Faraday's Law and the absence of magnetic monopoles. This is just due to the symmetry properties of the curvature tensor as we do not have torsion, ie. the Christoffel connections enjoy symmetry.

We also require that $\nabla^\mu J_\mu = 0$, which is the familiar conservation of charge statement. For us it means that the divergence of electromagnetic four-potential must obey the covariant wave equation

$$\Box\phi = 0, \tag{18}$$

where $\phi = \nabla^\mu A_\mu$ is the covariant divergence.

Conclusions

- Electromagnetism is induced by the twisting geometry of the spacetime. As the metric tensor $g_{\mu\nu} = A_\mu A_\nu$ depends solely on the electromagnetic four-potential, Ricci flatness requires that Maxwell's equations are satisfied.
- The classical action for electrodynamics is understood through minimizing rotation and preferring the level sets of divergence.
- The classical action of electromagnetism is actually the Einstein-Hilbert Action and electromagnetism can be understood therefore from the frameworks of General Relativity.

- Lindgren, J., Liukkonen, J. Quantum Mechanics can be understood through stochastic optimization on spacetimes. Sci Rep 9, 19984 (2019). https://doi.org/10.1038/s41598-019-56357-3
- G.Y. Rainich, Electrodynamics in General Relativity, Trans. Amer. Math. Soc. ,27, 106 (1925)
- C.W. Misner, J.A. Wheeler,Classical Physics as Geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. Ann. of Phys.,2, 525-603 (1957)
- Einstein, A., The Meaning of Relativity, 1922, The Hebrew University of Jerusalem

First, we show that the vacuum Einstein equation is the optimality equation for a certain coordinate-invariant functional. The object we are interested in is the total curvature of the Lorentzian manifold. This can be defined through a 4-rank curvature tensor, called the Riemann-Christoffel tensor. This tensor can be contracted and the result is the Ricci curvature tensor $R_{\mu\nu}$. According to Albert Einstein (1879-1955), Ricci curvature is essentially describing the local (mean-) curvature of spacetime, and the source of essentially this curvature is the mass and energy distribution (the symmetric stress-energy tensor). There is a natural invariant, called the scalar curvature R, which is the trace of the Ricci tensor. Minimizing this scalar curvature over invariant volume forms leads to the famous Einstein field equations of General Relativity. David Hilbert (1862-1943) apparently discovered this already in 1915. The key point we want to demonstrate here is that the criticality condition is independent of the sign of the metric determinant g.

We want to make the following functional stationary (without the source-term) :

$$S = \int R\sqrt{g}d^4x. \tag{19}$$

This functional is one of the simplest nontrivial curvature functionals. The inclusion of the metric determinant is due to the requirement of coordinate-invariance, as the metric tensor is the square of the Jacobian. Therefore, the determinant of the metric tensor is the determinant of the Jacobian determinant squared:

$$g = (detJ)^2. \tag{20}$$

Taking a square root gives: $\sqrt{g} = \sqrt{(detJ)^2}$ giving $|detJ| = \sqrt{g}$, so that the invariant volume form is:

$$dV = \sqrt{g}dx^4 \tag{21}$$

So that when minimizing scalar curvature it is reasonable to find a metric which indeed makes the following functional stationary:

$$S = \int R\sqrt{g}dx^4. \tag{22}$$

Functional variation (vary with respect to the contravariant metric tensor $g^{\mu\nu}$) gives:

$$\int \left(\delta R\sqrt{g} + R\delta\sqrt{g}\right) dx^4 = 0. \tag{23}$$

We have Jacobi's formula:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \tag{24}$$

The second term is interesting so let us focus on it:

$$\int \left(\delta R \sqrt{g} + \frac{1}{2} R \frac{\delta g}{\sqrt{g}} \right) dx^4 = 0. \tag{25}$$

Substituting then the variation δg we have:

$$\int \left(\delta R \sqrt{g} + \frac{1}{2} R \frac{g g^{\mu\nu} \delta g_{\mu\nu}}{\sqrt{g}} \right) dx^4 = 0. \tag{26}$$

$$\int \left(\delta R \sqrt{g} + \frac{1}{2} R \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} \right) dx^4 = 0. \tag{27}$$

$$\int \left(\delta R + \frac{1}{2} R g^{\mu\nu} \delta g_{\mu\nu} \right) \sqrt{g} dx^4 = 0. \tag{28}$$

$$\int \left(\delta R - \frac{1}{2} R g_{\mu\nu} \delta g^{\mu\nu} \right) \sqrt{g} dx^4 = 0. \tag{29}$$

Remembering that $R = R_{\mu\nu} g^{\mu\nu}$, from which we can conclude that we have the Einstein equation in vacuum:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{30}$$

It is important to note, which is indeed one of key points here, that the stationarity equation is the Einstein field equation in vacuum irrespective of the sign of the metric determinant. Therefore whenever we optimize the scalar curvature over invariant volumes, the metric obeys the nonlinear Einstein field equation above. The complete technical argument for the variation of the Ricci scalar can be found in any good textbook on General Relativity.