Towards modeling the Maxwell radiation field in an axially symmetric galaxy

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The 12th Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields
1–4 June 2020 — Virtual Meeting Online

Equation numbers in text, & references (in green), are clickable.

Motivation

Check the following prediction of an alternative theory of gravity, the "scalar ether theory" (MA, Braz. J. Phys. **36**, 177, 2006):

In the presence of both a gravitational field & an electromagnetic (EM) field, there must appear some "interaction energy", which should be distributed in space, and be gravitationally active (MA, Open Phys. **16**, 488, 2018).

That energy could thus possibly contribute to the dark matter. It is determined by the gravitational and EM fields, and by the velocity of the reference frame used w.r.t. the preferred frame of the theory.

Motivation (continued)

Goal of the present work: to build a representative analytical model for the EM field in a galaxy — especially for the *spatio-temporal variation* of the field.

This is in order to be able later to calculate the interaction energy (and check if its distribution has something common with a dark matter halo), but the present goal is interesting *per se*.

The field has to be an exact solution of the Maxwell equations in free space.

Main assumptions of the model

(i) EM field considered = sum of the fields emitted by the stars.
More exactly, the "shape" of the field is that of the sum field.
The magnitude of the field doesn't have to be that of the sum field.

(ii) The distribution of the stars is axially symmetric,
hence so is also the EM field (except for peculiar cases, and except for the vicinity of a star).

(iii) Each star emits a spherical radiation.

DISTRIBUTION OF THE STARS: PRINCIPLE

- Each star seen as a point determined by cylindrical coordinates (ρ, z, ϕ): distance to the symmetry axis, altitude, azimuth.

- A discrete set $\mathsf{G} = \{(\rho_i, z_j, \phi_k)\}$ by random generation.

Distribution of the stars: details

- Exponential distribution for ρ:

$$P(a < \rho < b) = \frac{1}{h} \int_a^b e^{-\frac{\rho}{h}} \, d\rho, \tag{1}$$

with $h = 3\,\mathrm{kpc}$ (Milky Way) in the numerical computations.

- Exponential distribution for $z > 0$ at given ρ, with $h_z = 0.2\,\mathrm{kpc}$.

- Symmetry w.r.t. $z = 0$: for each star at (ρ_i, z_j, ϕ_k) with $z_j > 0$, another one at $(\rho_i, -z_j, \phi_k)$.

- Uniform distribution of ϕ at given ρ and z ($\Rightarrow$ axial symmetry).

AXIALLY SYMMETRIC SCALAR WAVES

A) Any *time-harmonic* axisymmetric solution of the *scalar* wave eqn can be written in the form (Zamboni-Rached *et al.* 2008)

$$\psi_{\omega\ S}(\rho, z, t) = e^{-\mathrm{i}\omega t} \int_{-\omega/c}^{\omega/c} J_0\left(\rho\sqrt{\frac{\omega^2}{c^2} - k^2}\right) e^{\mathrm{i}\,k\,z}\, S(k)\, \mathrm{d}\,k, \tag{2}$$

with J_0 the (first-kind) Bessel function of order 0.
(This result applies to totally-propagating solutions, i.e. without evanescent components.)

B) The *general* axisymmetric solution of the scalar wave eqn is got by considering a frequency spectrum. With a discrete frequency spectrum:

$$\Psi_{(\omega_j)\ (S_j)}(\rho, z, t) = \sum_j \psi_{\omega_j\ S_j}(\rho, z, t). \tag{3}$$

Axially symmetric EM fields

Let an axisymmetric solution Ψ of the scalar wave eqn be given. Passing through its time-harmonic components, one may associate with it two exact solutions of the free Maxwell eqs:

(i) a solution of the form (B_ϕ, E_ρ, E_z) with $E_\phi = B_\rho = B_z = 0$.

(ii) a solution of the form (E'_ϕ, B'_ρ, B'_z) with $B'_\phi = E'_\rho = E'_z = 0$ — by EM duality from the first solution, i.e.,

$$\mathrm{E}' = c\mathrm{B},\ \ \mathrm{B}' = -\mathrm{E}/c. \tag{4}$$

The first solution derives from the following vector potential A:

$$\mathrm{A} := \Psi \mathrm{e}_z, \qquad \text{or} \quad A_z := \Psi,\ \ A_x = A_y = 0. \tag{5}$$

Axially symmetric EM fields (continued)

For each time-harmonic component, using the free Maxwell eqs plus the harmonic time dependence gives $\mathbf{E}, \mathbf{B}$ for the first solution (Garay-Avendaño & Zamboni-Rached, Appl. Opt. **53**, 4524, 2014):

$$B_{\phi\,\omega} = -\frac{\partial A_{z\,\omega}}{\partial \rho}, \qquad E_{\phi\,\omega} = 0, \tag{6}$$

$$E_{\rho\,\omega} = \mathrm{i}\,\frac{c^2}{\omega}\,\frac{\partial^2 A_{z\,\omega}}{\partial \rho\,\partial z}, \qquad B_{\rho\,\omega} = 0, \tag{7}$$

$$E_{z\,\omega} = \mathrm{i}\,\frac{c^2}{\omega}\,\frac{\partial^2 A_{z\,\omega}}{\partial z^2} + \mathrm{i}\,\omega A_{z\,\omega}, \qquad B_{z\,\omega} = 0. \tag{8}$$

NB: As usual, it is implicit that, in Eqs. (6)–(8), B_ϕ, E_ρ and E_z are actually the *real parts* of the respective r.h.s.

Axially symmetric EM fields (end)

Somewhat surprisingly, it turns out that these two classes of solutions generate *all* axisymmetric solutions of the free Maxwell eqs. First, in the time-harmonic case, we have:

Theorem. *Let* $(\mathrm{A}, \mathrm{E}, \mathrm{B})$ *be any time-harmonic axisymmetric solution of the free Maxwell equations. There exist a unique solution* $(\mathrm{E}_1, \mathrm{B}_1)$ *of the first class (6)–(8) and a unique solution* $(\mathrm{E}'_2, \mathrm{B}'_2)$ *of the second class, both with the same frequency as has* $(\mathrm{A}, \mathrm{E}, \mathrm{B})$*, and whose sum gives just that solution:*

$$\mathrm{E} = \mathrm{E}_1 + \mathrm{E}'_2, \qquad \mathrm{B} = \mathrm{B}_1 + \mathrm{B}'_2. \tag{9}$$

(MA, Open Phys., 2020, to appear, and HAL preprint hal-02434217).

The general axisymmetric case follows by summation on frequencies. (Ibidem).

The case of spherical waves

The case of a *spherical* time-harmonic solution of the scalar wave eqn is obtained by putting $S(k) \equiv \frac{c}{2\omega} \quad (-\frac{\omega}{c} < k < \frac{\omega}{c})$ in the axisymmetric time-harmonic solution (2):

$$\psi_{\omega \ S \equiv \frac{c}{2\omega}} (\rho, z, t) = e^{-\mathrm{i}\omega t} \operatorname{sinc}\left(\frac{\omega}{c} R\right), \tag{10}$$

where $\operatorname{sinc}\theta := \frac{\sin\theta}{\theta}, \quad R := |\mathrm{x}| = \sqrt{\rho^2 + z^2}$.

For a set of spherical sources situated at the points x_i, each source having the same frequency spectrum (S'_j, ω_j), we thus have:

$$\Psi_{(\mathrm{x}_i)\ (\omega_j)\ (S'_j)} (\rho, z, t) = \sum_i \sum_j S'_j \, e^{-\mathrm{i}\omega_j t} \operatorname{sinc}\left(\frac{\omega_j}{c} R_i\right), \tag{11}$$

where $R_i := |\mathrm{x} - \mathrm{x}_i|$, and setting again the initial phases to zero for simplicity.

THE MODEL

Step **1**. Determine a relevant axisymmetric solution of the scalar wave eqn, by fitting to the form (2)–(3) the sum (11) of the spherical radiations emitted by the "stars" that make the "galaxy".

Step **2**. Calculate the associated EM field of the first class $(E_\rho, E_z, B_\phi\,; B_\rho = B_z = E_\phi = 0)$, by summing the time-harmonic contributions given by Eqs. (6)–(8).

Similarly, calculate the associated EM field of the second class $(B'_\rho, B'_z, E'_\phi\,; E'_\rho = E'_z = B'_\phi = 0)$ by using the EM duality (4).

(One may consider two different frequency spectra for the two classes, e.g. "mutually interpenetrating" ones.)

Step **1** (especially) is delicate numerically, as we now show.

PRECISE OBJECT OF THE FITTING

The fitting is done to determine the "wave vector spectra" S_j in Eq. (3):

$$\Psi_{(\omega_j)\ (S_j)}(\rho, z, t) = \sum_j \psi_{\omega_j\ S_j}(\rho, z, t) \tag{3}$$

where [Eq. (2) with $\omega = \omega_j$]

$$\psi_{\omega_j\ S_j}(\rho, z, t) = e^{-\mathrm{i}\,\omega_j t} \int_{-\omega_j/c}^{\omega_j/c} J_0\left(\rho\sqrt{\frac{\omega_j^2}{c^2} - k^2}\right)\ e^{\mathrm{i}\,k\,z}\ \underline{S_j(k)}\ \mathrm{d}\,k. \tag{12}$$

Thus, one S_j for each value of j (that specifies the frequency ω_j). Here $S_j = S_j(k)$, with $k = k_z$ the projection of the wave vector on the z axis (the symmetry axis).

THE MAIN DIFFICULTY

The difficulty comes from the huge ratio

$$\frac{\text{galactic distances}}{\text{wavelength}} \simeq \frac{\text{kpc}}{\mu\text{m}} \simeq 3 \times 10^{25}. \tag{13}$$

This huge number discards several possibilities. E.g., determining S_j by its Fourier coefficients — as proposed (for a very different problem), by Garay-Avendaño & Zamboni-Rached (Appl. Opt. **53**, 4524, 2014). This turns out to be not tractable at all here.

Instead, we determine S_j by the values $S_{nj} := S_j(k_{nj})$, where $k_{nj} := -K_j + 2nK_j/N$ $(n = 0, ..., N)$ is a regular discretization of the integration interval $[-K_j, +K_j]$ for frequency ω_j, Eq. (12). Integrals like the one in Eq. (12) are approximated by discrete sums (composite Simpson 3/8 rule):

$$\int_{-K_j}^{+K_j} f(k)\,\mathrm{d}\,k \simeq \sum_{n=0}^{N} a_{nj} f(k_{nj}) + O\left(\frac{1}{N^4}\right). \tag{14}$$

Double precision is far from enough

The ratio in Eq. (13), thus $\mathcal{O}(10^{25})$, gives the magnitude of the argument of the sinc function in Eq. (11) and the argument of the Bessel function J_0 in Eq. (12).

However, the sinc and Bessel J_0 functions oscillate around 0 with a pseudo-period $\mathcal{O}(1)$. So already to get only the correct *sign*, one needs to know their arguments to a precision better than $\mathcal{O}(1)$.

In view of the magnitude of the arguments: $\mathcal{O}(10^{25})$, it means: 25 significant digits needed to know just the sign of sinc & J_0.

Double precision is far from enough (continued)

Therefore, double precision (16 significant digits) is not enough at all: at least quadruple precision (32 significant digits) is needed — and even, it is not a luxury.

Implementing many-digits precision, using the Matlab function vpa, increases drastically the computation time.

We use the external toolbox “Multiprecision Computing Toolbox for Matlab”, of Advanpix, which is much quicker.

CORRECTNESS TEST

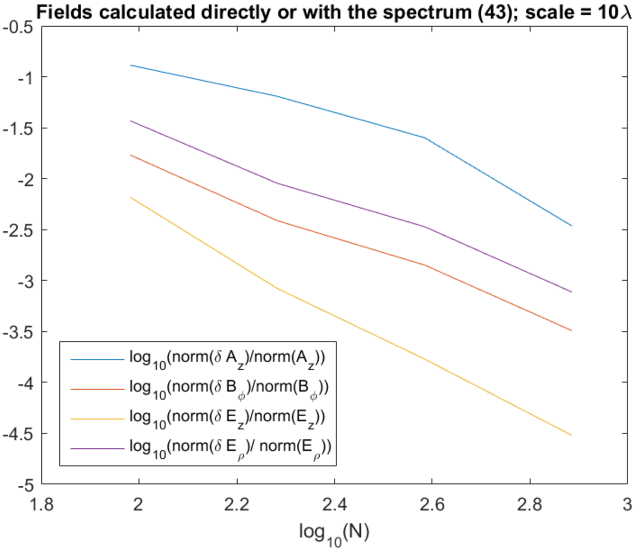
Starting from Eq. (3) with (12), the scalar potential $A_z = \Psi$, as well as the EM field deduced from it by Eqs. (6)–(8), involve integrals which are calculated as discrete sums, Eq. (14).

To check the implementation of these calculations, we use the spherical case. An exact solution of the free Maxwell equations is got by Eqs. (6)–(8), with $A_z = \psi_\omega \;{}_{S\equiv\frac{c}{2\omega}}$ the spherically-symmetric time-harmonic solution (10) of the scalar wave equation.

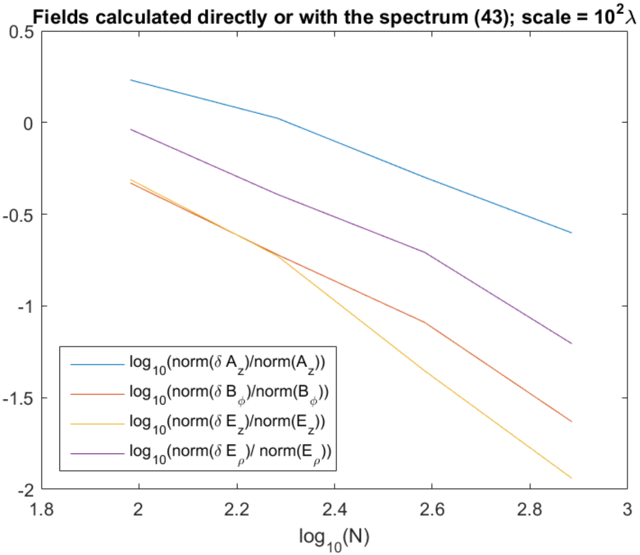
The next three pages show the normed average quadratic differences between the fields $\Psi = A_z,\ B_\phi,\ E_\rho,\ E_z$, as calculated either "directly", i.e., as just explained, or "with the spectrum (43)", i.e., as explained on the top, and using here the discretized constant spectrum $S \equiv \frac{c}{2\omega}$:

$$S_n := S(k_n) = \frac{c}{2\omega} \quad (n = 0, ..., N). \qquad \text{"(43)"}$$

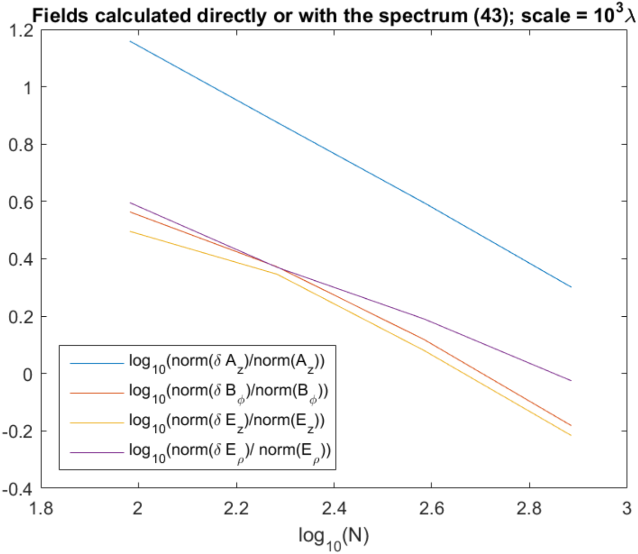
Errors vs number N of integration intervals (1)



ERRORS VS NUMBER N OF INTEGRATION INTERVALS (2)



ERRORS VS NUMBER N OF INTEGRATION INTERVALS (3)



Correctness test (end)

As expected, the errors (the relative δ 's on the different fields calculated either "directly" or "with the spectrum (43)") decrease strongly as the discretization of the (here constant) spectrum function $S(k)$ becomes finer, i.e., with increasing N.
<u>This validates the correctness of our calculations.</u>

However, the errors increase when the scale length scale is increased, here from 30 m to 3 km (here $\lambda = 3$ m). Reason: The integrals in Eq. (12) and in the eqs. for B_ϕ, E_ρ, E_z involve functions of k that oscillate with a frequency or a pseudo-frequency which is proportional to the magnitude of the spatial variables ρ and z. These integrals being approximated by discrete sums as in Eq. (14), one would have to increase the discretization number N accordingly.

Numbers used

A file of $\simeq 10^4$ randomly generated "stars" used here.

11 frequencies centered at $\omega_0 \equiv \frac{2\pi c}{\lambda_0}$ with $\lambda_0 \equiv 0.5\,\mu\mathrm{m}$ (center of visible spectrum), having Gaussian weights with $\sigma = \frac{\omega_0}{2}$.

Various discretization grids tried for the spacetime domain (variables ρ, z, t) at the stage of *fitting*. Adopted sizes of that domain ($\rho = 0$ avoided: eqs for direct calculation don't apply):

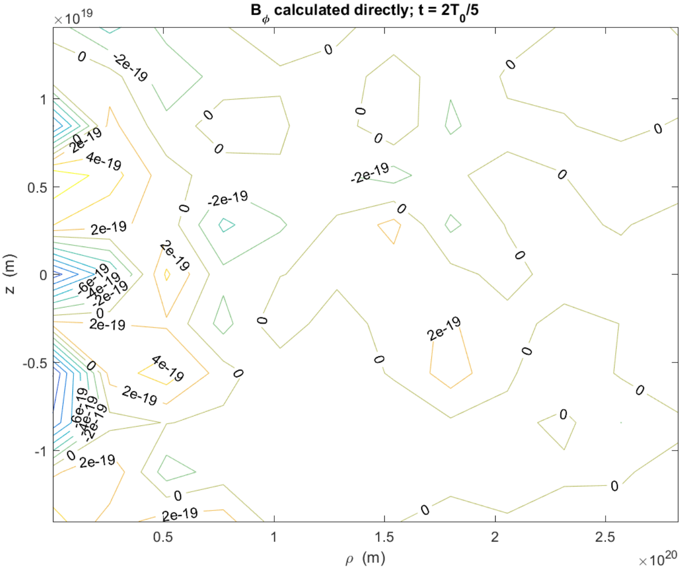
$$3 \times 10^{14}\,\mathrm{m} \leq \rho \leq 10\,\mathrm{kpc} \simeq 3 \times 10^{20}\,\mathrm{m},$$

$$-0.5\,\mathrm{kpc} \leq z \leq +0.5\,\mathrm{kpc} \simeq 1.5 \times 10^{19}\,\mathrm{m},$$

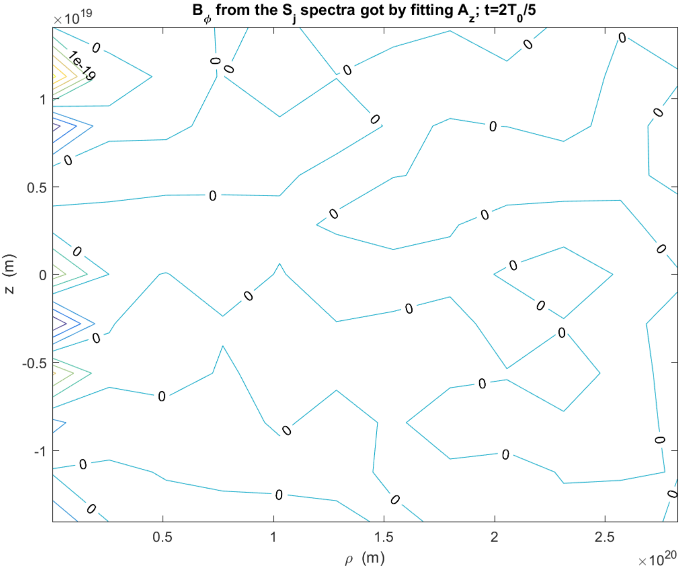
$$0 \leq t \leq T_0 \equiv \frac{\lambda_0}{c} .$$

For this fitting: Discretization number for k: $N = 48$. Number of spacetime points: $(N_\rho = 12) \times (N_z = 11) \times (N_t = 5)$.

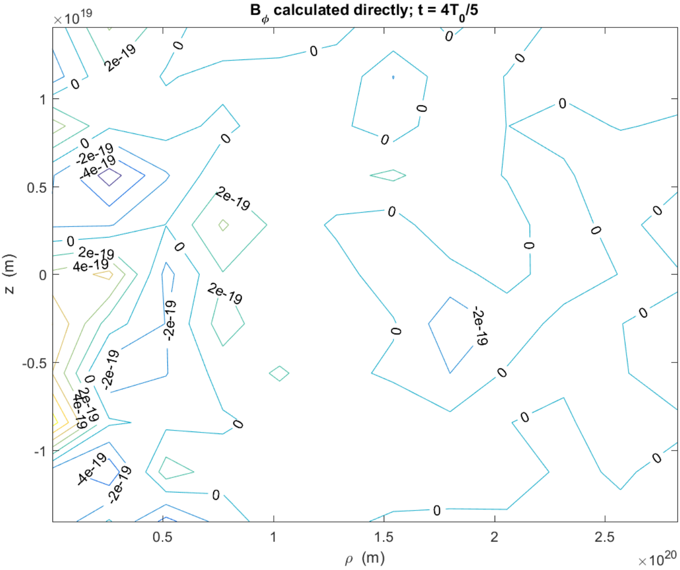
MAGNETIC FIELD B_ϕ FROM DIRECT CALCULATION



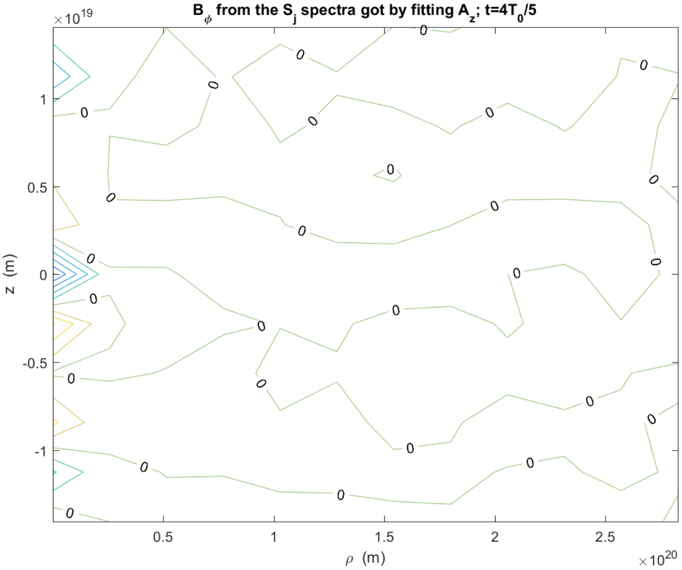
MAGNETIC FIELD B_ϕ FROM THE MODEL



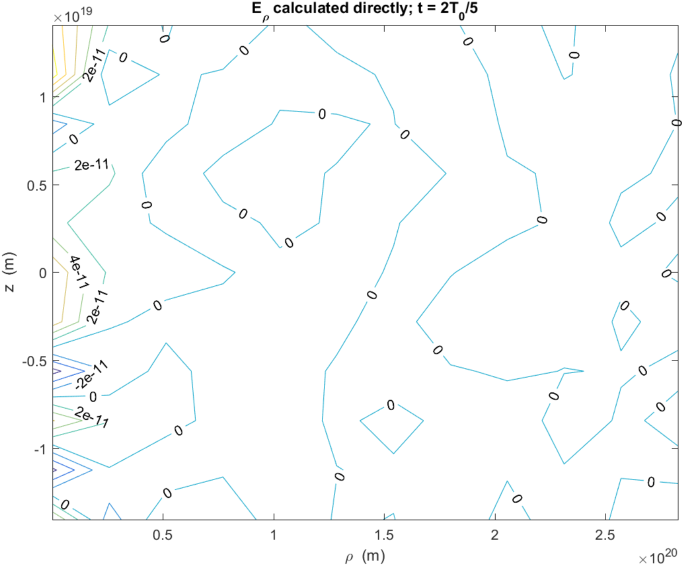
Magnetic field B_ϕ from direct calculation



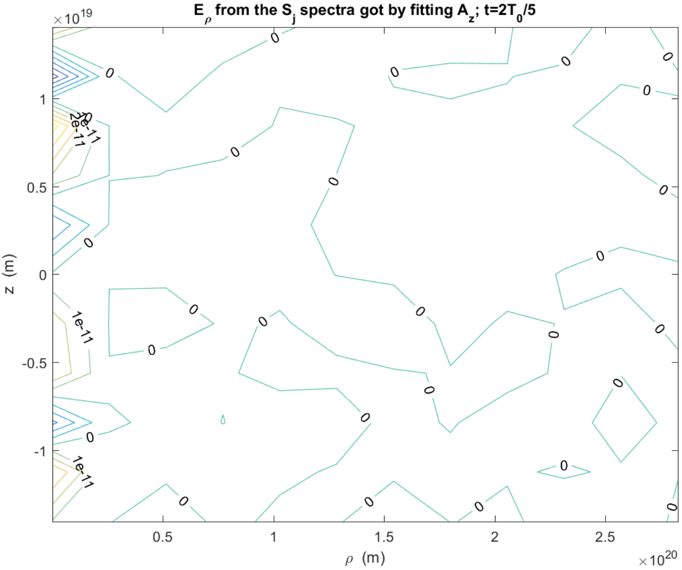
MAGNETIC FIELD B_ϕ FROM THE MODEL



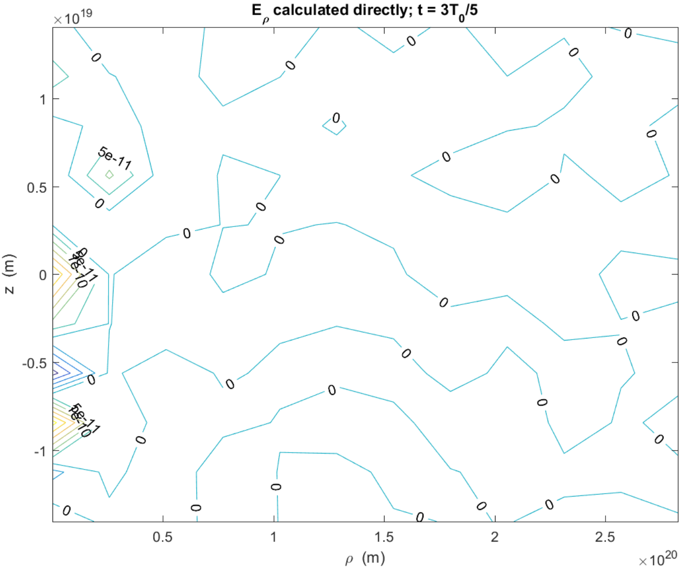
RADIAL ELECTRIC FIELD E_ρ FROM DIRECT CALCULATION



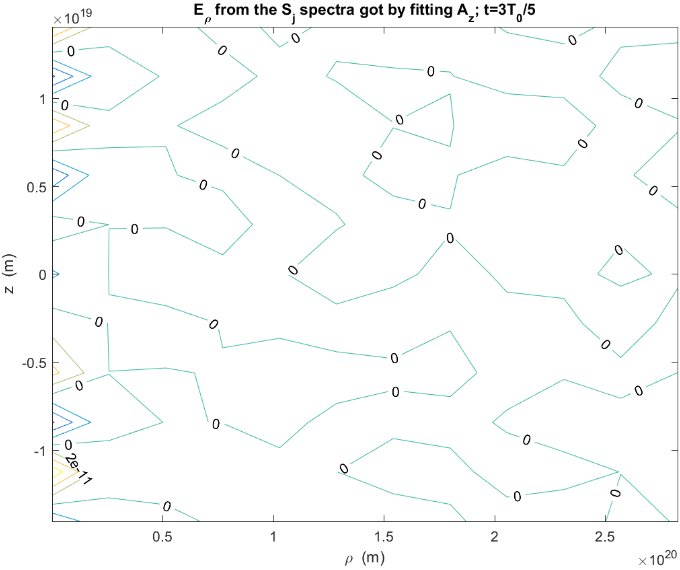
RADIAL ELECTRIC FIELD E_ρ FROM THE MODEL



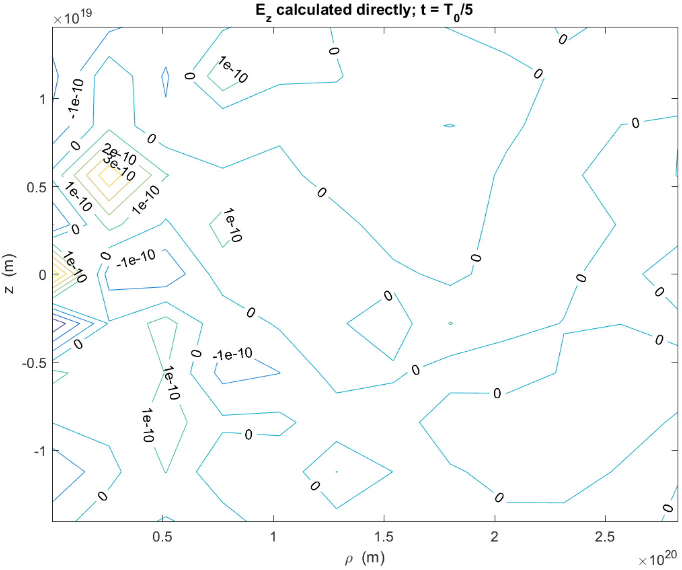
RADIAL ELECTRIC FIELD E_ρ FROM DIRECT CALCULATION



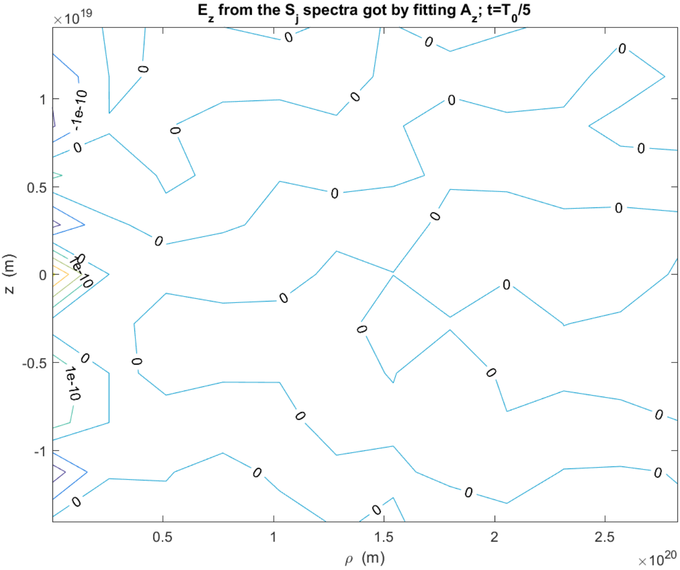
RADIAL ELECTRIC FIELD E_ρ FROM THE MODEL



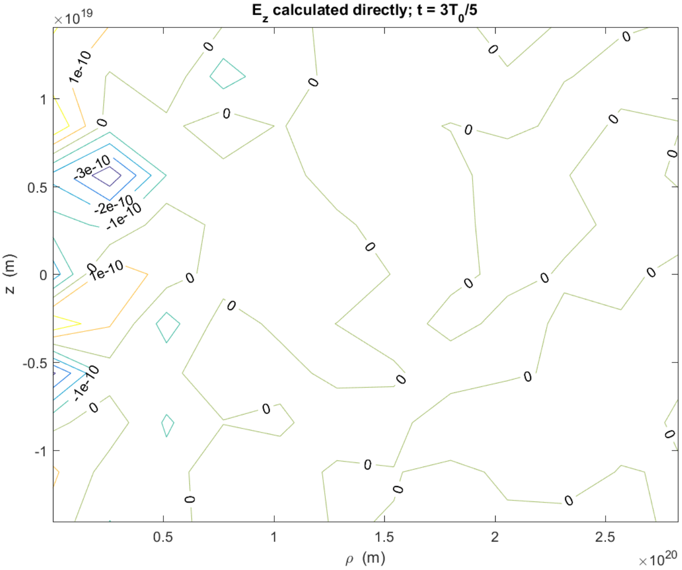
AXIAL ELECTRIC FIELD E_z FROM DIRECT CALCULATION



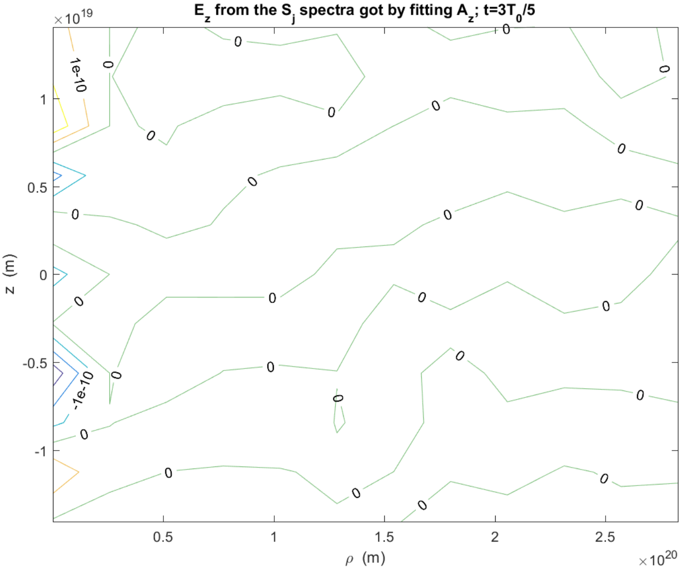
AXIAL ELECTRIC FIELD E_z FROM THE MODEL



Axial electric field E_z from direct calculation



Axial electric field E_z from the model



Comments on the results

The increase of the error with scale, found for the smaller scales investigated here above (up to $10^3\lambda$), fortunately does not continue up to the scale relevant to a typical disc galaxy ($\simeq 3 \times 10^{25}\lambda$). This is likely because the spectra are now obtained by fitting.

For the present calculation, the normed quadratic errors (e.g. $\parallel \delta B_\phi \parallel / \parallel B_\phi \parallel$) are close to unity. The qualitative features of the fields are the same for the fields calculated directly or with the model, i.e., from the spectra got by fitting the A_z potential: the fields are more intense close to the z axis (although this feature is somewhat accentuated with the model), and the maximum intensities (positive and negative) have very similar values.

It seems like the fields got from the model are as representative of a galaxy as the fields calculated directly can be.

CONCLUSION

- Analytical model for Maxwell field in an axisymmetric galaxy.
- Based on an explicit representation for axisymmetric free Maxwell fields. General applicability of this representation has been proved.
- Model adjusted by fitting to it the sum of spherical radiations emitted by the composing "stars".
- Huge ratio distance/wavelength needs precision > quadruple.
- Model has passed some tests. Results indicate fields are highest near to z axis, and E_z slightly dominates over E_ρ.